Almost Optimal Time Lower Bound for Approximating Parameterized Clique, CSP, and More, under ETH

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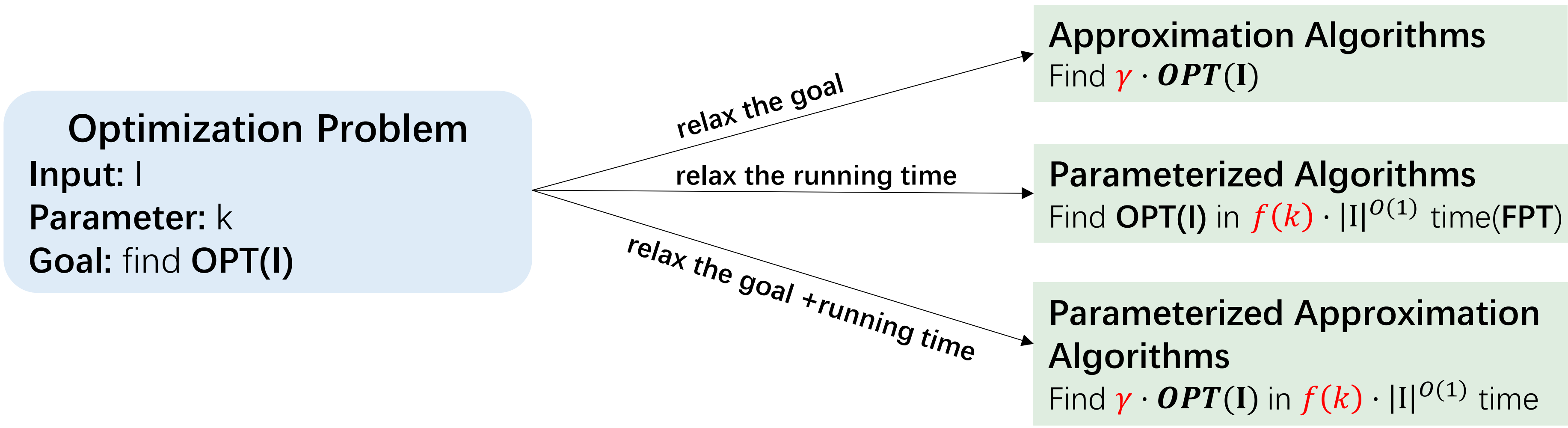
Outline

- **Introduction**
- Proof Sketch
- Conclusion



Algorithmic results:
Min-k-Cut[GLL18b, GLL18a, KL20, LSS20], **k-Clustering**[ABB+23], **k-Means/k-Median**[CGTS02, KMN+04, LS16, BPR+17, CGK+19, ANSW20], **Vertex-Coloring**[DHK05, Mar08], **k-Path-Deletion**[Lee19]

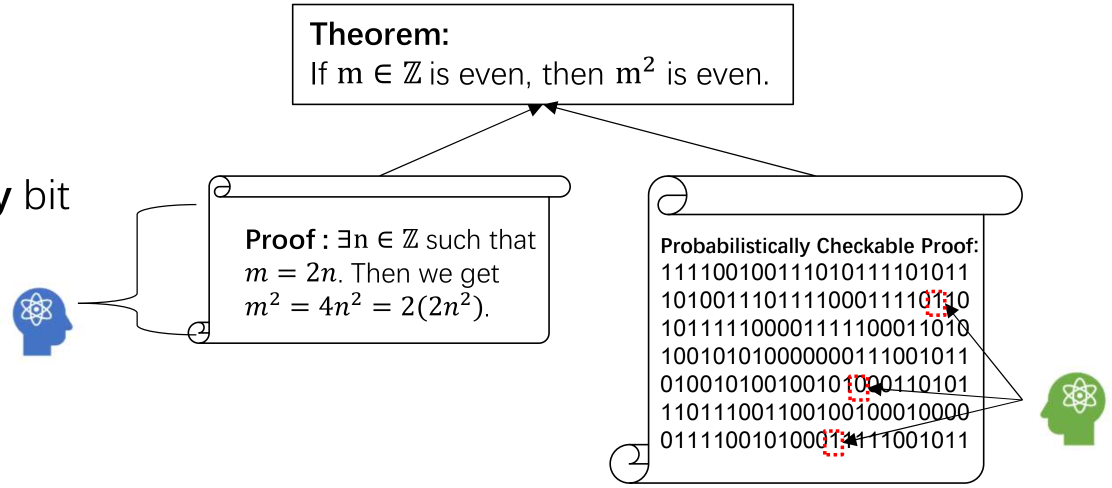
Hardness results:
k-SetCover[CHK13,CCK+17, CL19, KLM19, Lin19, KN21, LRSW23a], **k-Clique**[CHK13, CCK+17, Lin21, LRSW22, KK22, CFLL23, LRSW23b], **k-Steiner Orientation**[Wło20],**Max-k-Coverage**[Man20], **k-Set-Intersection**[Lin18, BKN21], **k-Min-Distance-Code**[Man20, BBE+21, BCGR23]

PCP Theorem for parameterized complexity?

Probabilistically Checkable Proof

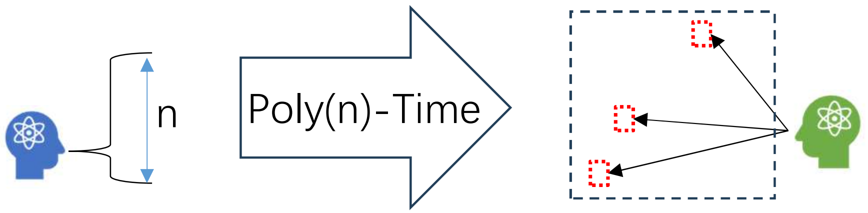
Naïve Verifier：
- needs to check **every** bit

PCP Verifier:
- randomly read **3** bits
- $\mathbf{1-\epsilon}$ error probability



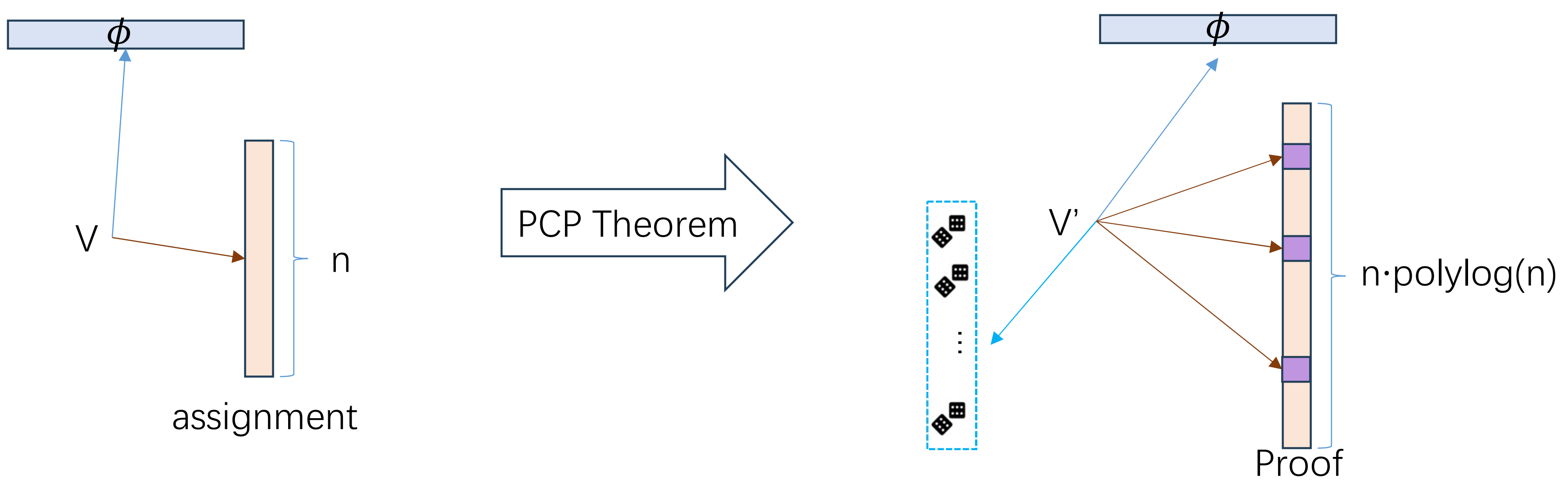
PCP Theorem(informal):

A polynomial time algorithm to convert a Naïve verifier to a PCP verifier.



PCP Theorem: Proof View

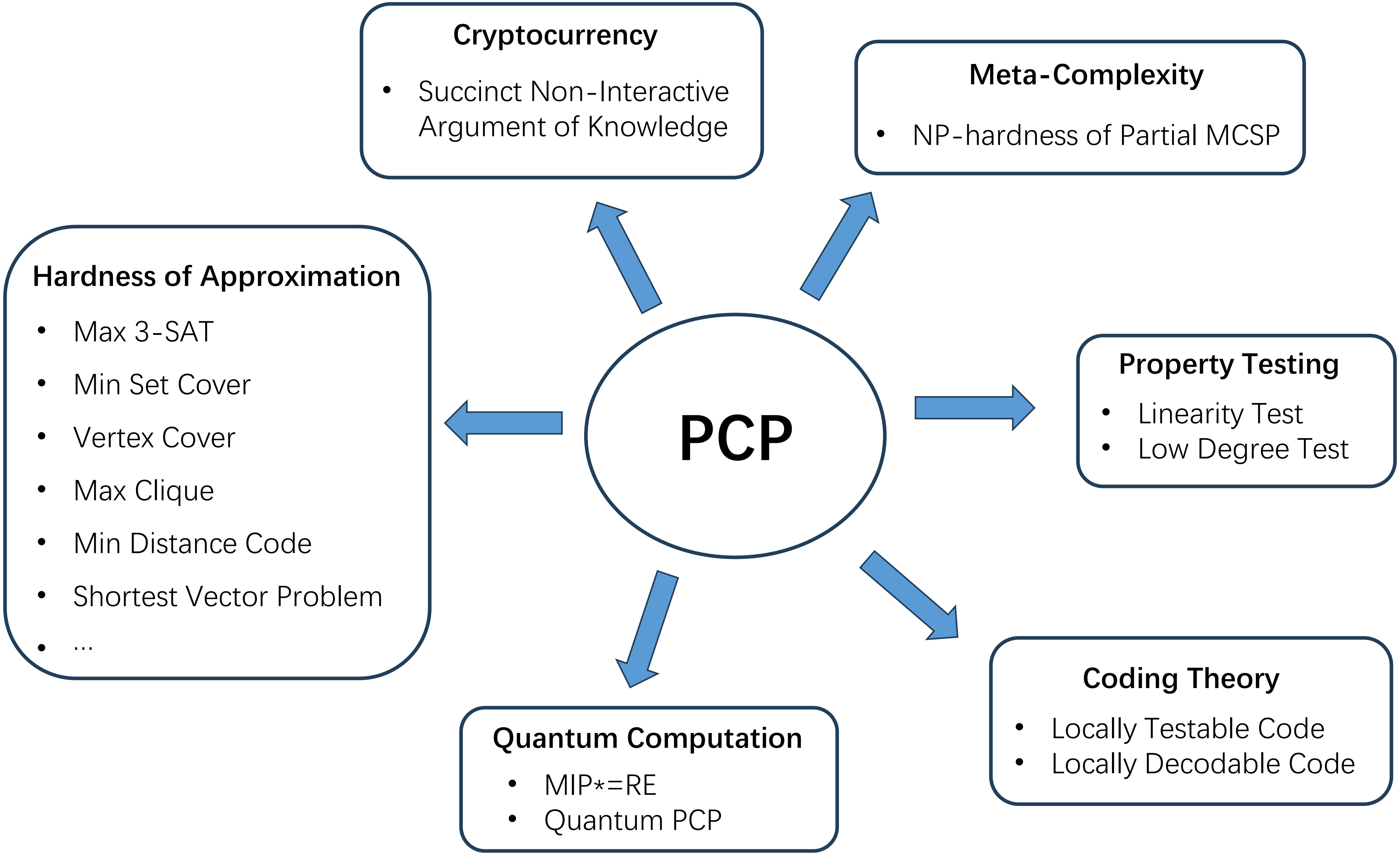
PCP Theorem: 3SAT ∈ $\mathrm{PCP}_{1,1-\varepsilon}[O(\log n), O(1)]_{\Sigma=O(1)}$



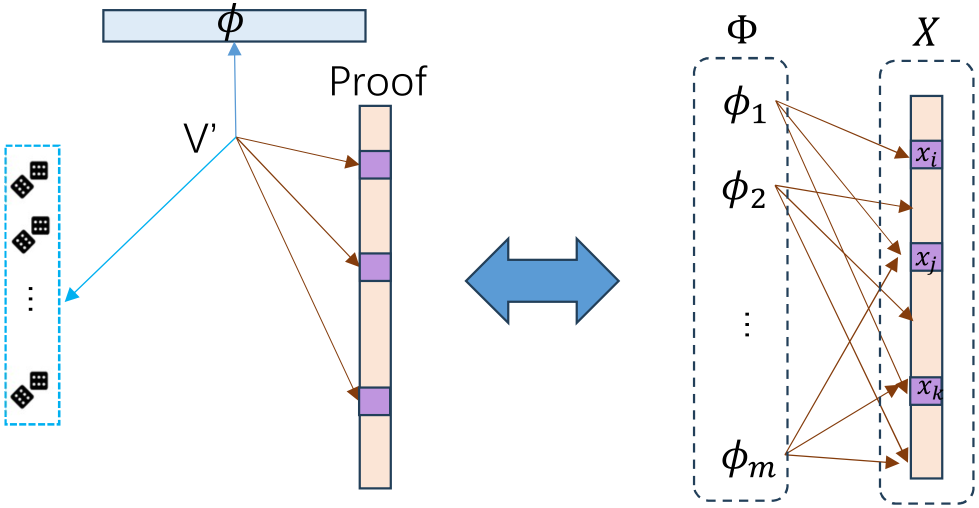
ϕ ∈ 3SAT if there exists a satisfying assignment

If $\phi \in$ 3SAT then ∃ proof $\Pr[V' accept] = 1$

If $\phi \notin$ 3SAT then ∀ proof $\Pr[V' accept] \leq 1 - \epsilon$


Cryptocurrency
• Succinct Non-Interactive Argument of Knowledge
Meta-Complexity
• NP-hardness of Partial MCSP
Hardness of Approximation
• Max 3-SAT
• Min Set Cover
• Vertex Cover
• Max Clique
• Min Distance Code
• Shortest Vector Problem
• ...
PCP
Property Testing
• Linearity Test
• Low Degree Test
Quantum Computation
• MIP*=RE
• Quantum PCP
Coding Theory
• Locally Testable Code
• Locally Decodable Code

PCP Theorem: Hardness of Approximation



If $\phi \in$ 3SAT then $\exists$ proof $\Pr[V'accept] = 1$

If $\phi \notin$ 3SAT then $\forall$ proof $\Pr[V'accept] \le 1 - \epsilon$

Constraint Satisfaction Problem

Input: $\Pi = (X, \Sigma, \Phi)$
- X: variables
- Σ: alphabet
- Φ: const-arity constraints

Question:
- $\exists \sigma: X \to \Sigma$ satisfying all constraints?

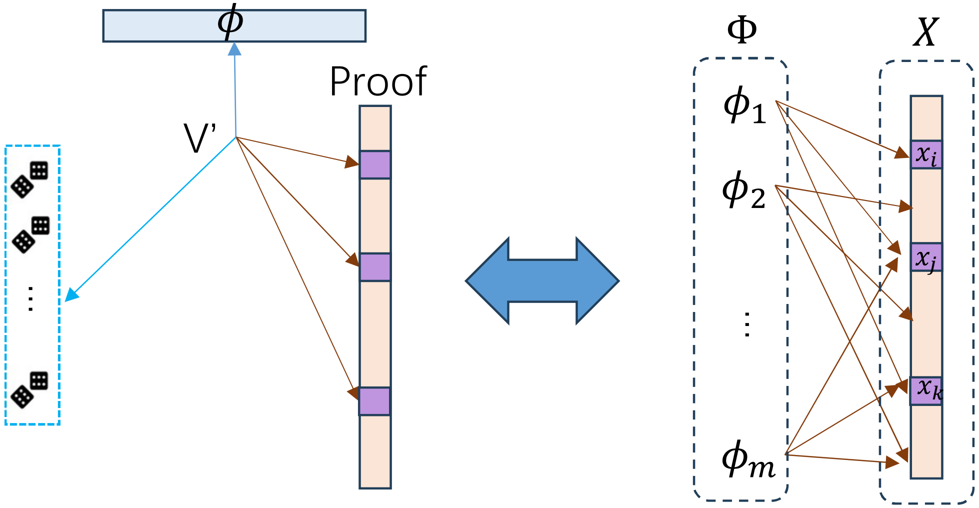
val(Π):=max. fraction of constraints satisfied by some assignment

$(1$ vs $\delta)$ gap CSP

Input: a CSP instance $\Pi = (X, \Sigma, \Phi)$
Goal: distinguish val(Π) = 1 vs val(Π)$\le \delta$

PCP Theorem:
For $\Sigma = O(1)$ and $|X| = n$, there is no $n^{O(1)}$ time algorithm for (1 vs 0.9) gap CSP assuming **P≠NP**.

Parameterized PCP: Hardness of Approximation



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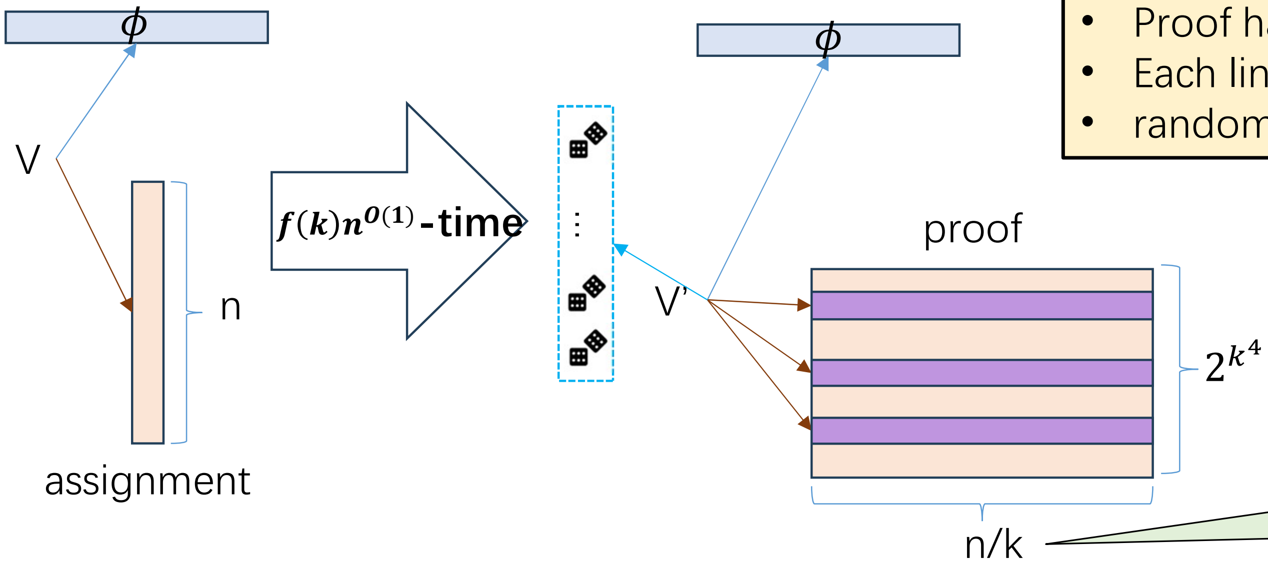
PIH (Parameterized Inapproximability Hypothesis) [Lokshtanov-Ramanujan-Saurabh-Zehavi'20]:
Let $k = |X|$ and $n = |\Sigma|$, there is no $f(k) \cdot n^{O(1)}$ time algorithm for (1 vs 0.9) gap parameterized CSP.

[GLRSW24]: ETH ⇒ PIH

ETH: n-variable 3SAT requires $2^{\Omega(n)}$-time

Parameterized PCP (Proof View):
- Proof has **f(k) lines**
- Each line has **n/g(k) bits**
- randomly read **O(1) lines**



Classical PCP:
O(n·**polylog(n)**/k)

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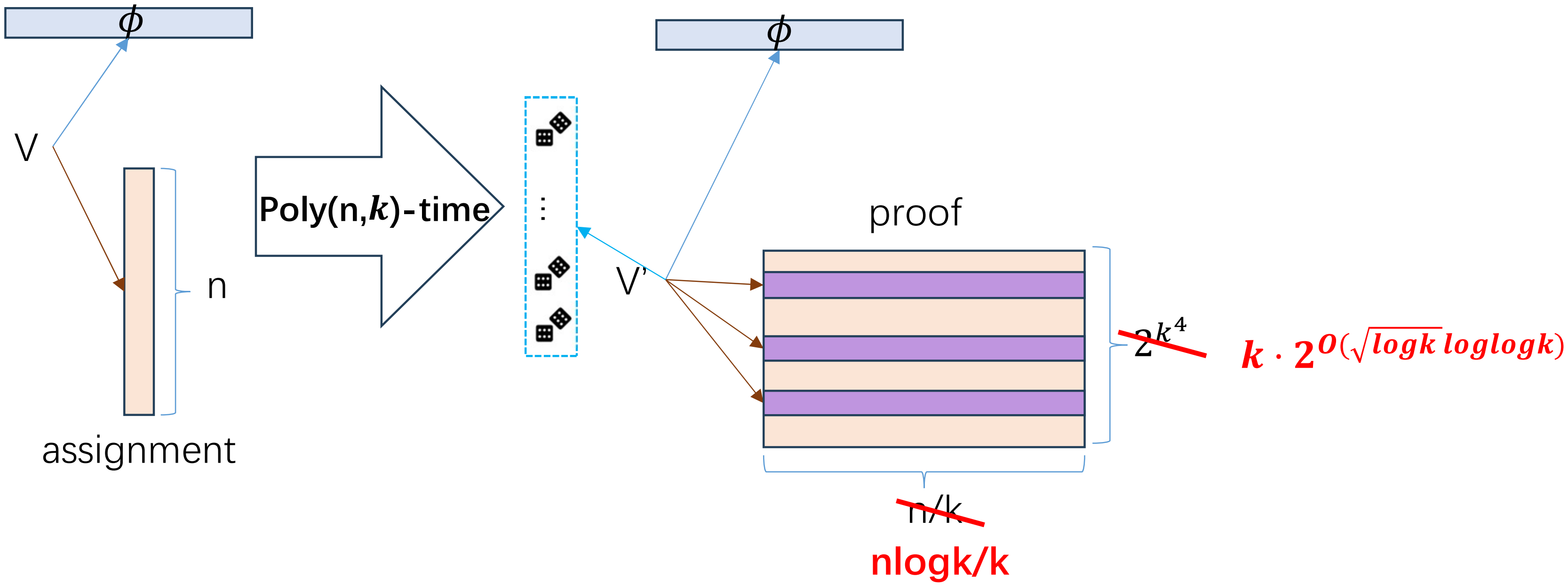
[This work]: short Parameterized PCP



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Applications

Problem	Assumption	Lower Bound	Hardness Approximation Ratio
k-variable n-alphabet CSP	**ETH**	no $f(k) \cdot n^{k^{1-o(1)}}$	Some $\epsilon \in (0,1)$
k-Clique	**ETH**	no $f(k) \cdot n^{k^{1-o(1)}}$	Any constant
Max-k-Coverage	**ETH**	no $f(k) \cdot n^{k^{1-o(1)}}$	Some $\epsilon \in (0,1)$
k-Exact-Cover	**ETH**	no $f(k) \cdot n^{k^{1-o(1)}}$	Some $\rho > 1$

Improved to $n^{\frac{k}{polylog k}}$ by
[Bafna, Karthik, Minzer STOC'25].

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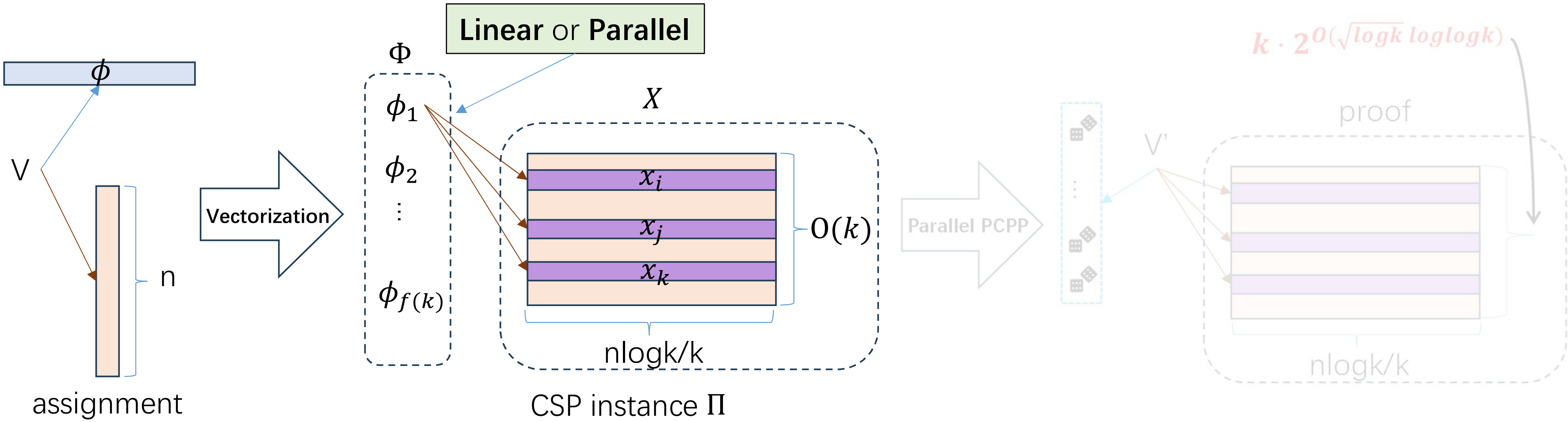
Proof Overview



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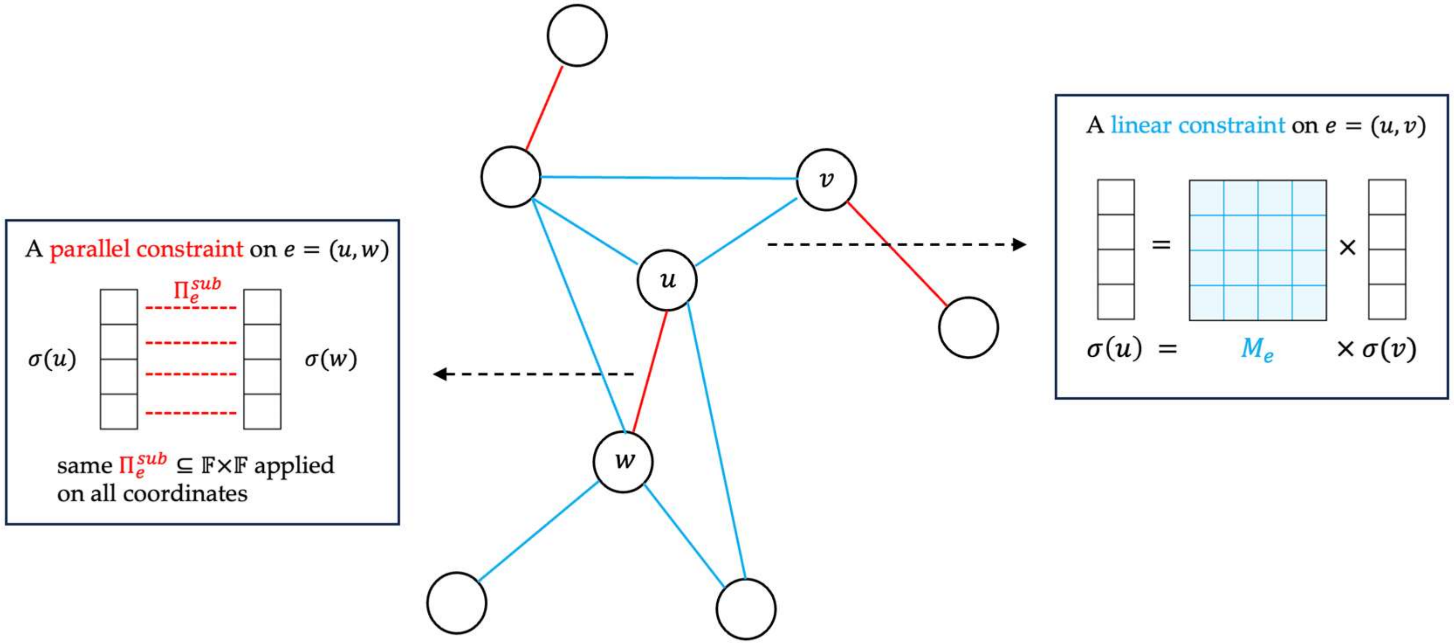
Vector-valued CSP

Vector-valued CSP
- **Alphabet：** vector space $\mathbb{F}^d$
- **Constraints：** divided into parallel part and linear part



3-SAT→VecCSP

Goal: Given an n-variable O(n)-clauses 3-CNF, construct an equivalent VecCSP

[GLRSW24]: $\mathbf{O(k^2)}$-variable $\boldsymbol{(n/k)}$-dimension VecCSP

[This work]: use results in [Mar10,KMPS23,CDNW25] to get $\mathbf{O(k)}$-variable $\boldsymbol{(\frac{n \cdot \log k}{k})}$-dimension VecCSP

Can You Beat Treewidth?[*]

Dániel Marx[†]

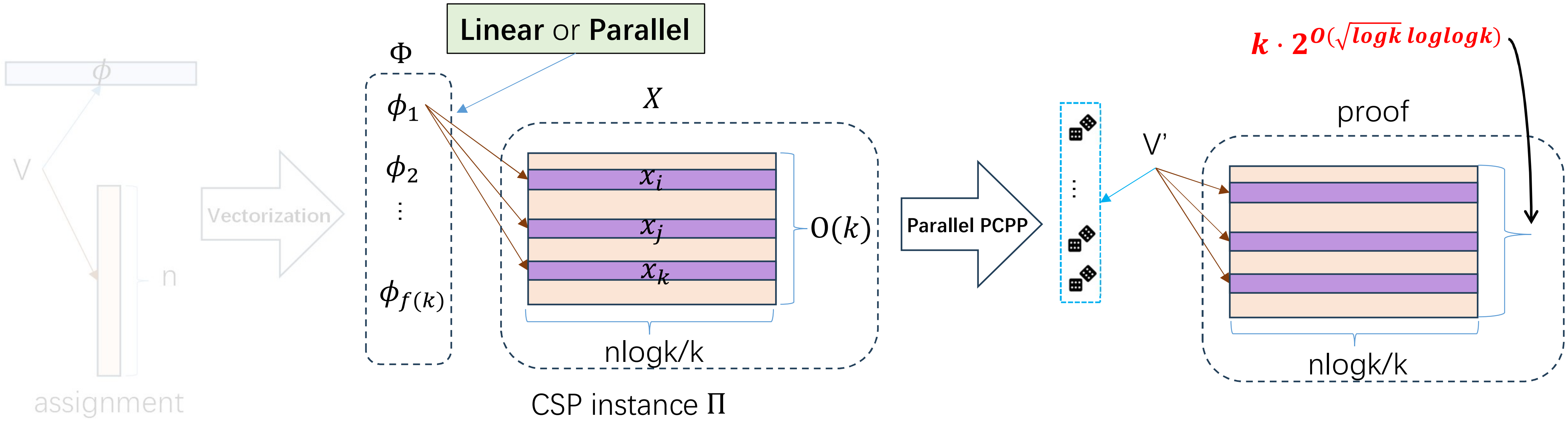
Conditional lower bounds for sparse parameterized 2-CSP:
A streamlined proof

Karthik C. S.[*] Dániel Marx[†] Marcin Pilipczuk[‡] Uéverton Souza[§]

Can You Link Up With Treewidth?

Radu Curticapean, Simon Döring, Daniel Neuen, Jiaheng Wang

Proof Overview



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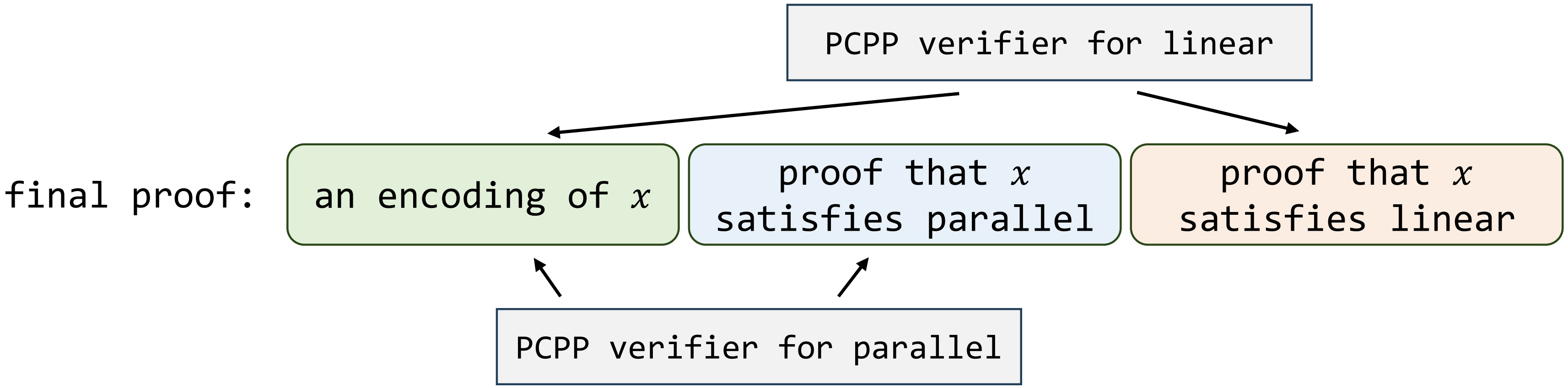
PCP of Proximity

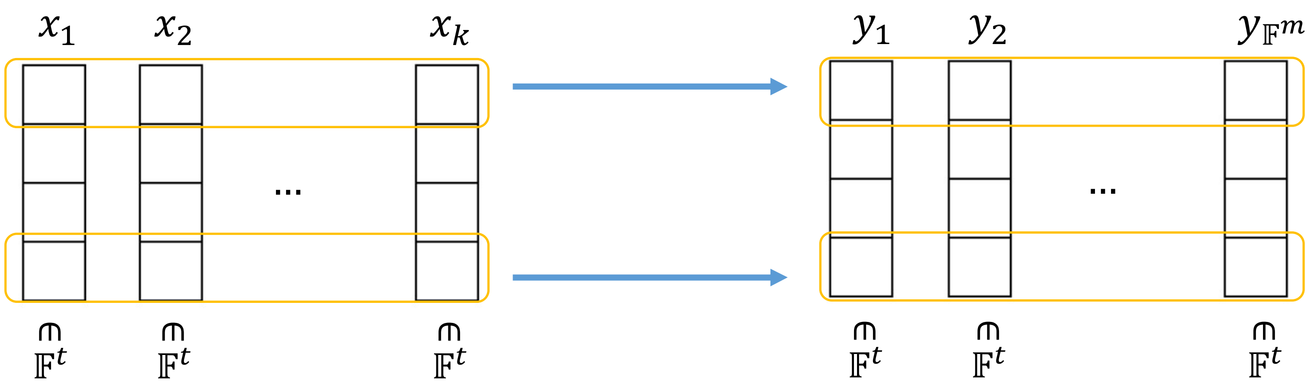
- Suppose Alice wants to convince you that a **VecCSP** instance Γ is satisfiable.
- She could give you a proof that the **parallel** part is satisfiable, and a proof that the **linear** part is satisfiable.
- Wait! How to ensure the two parts share **the same solution**?
- We need **PCP of proximity**! The statement to prove is not "**Γ is satisfiable**", but

"x is (the encoding of) a solution to Γ"!



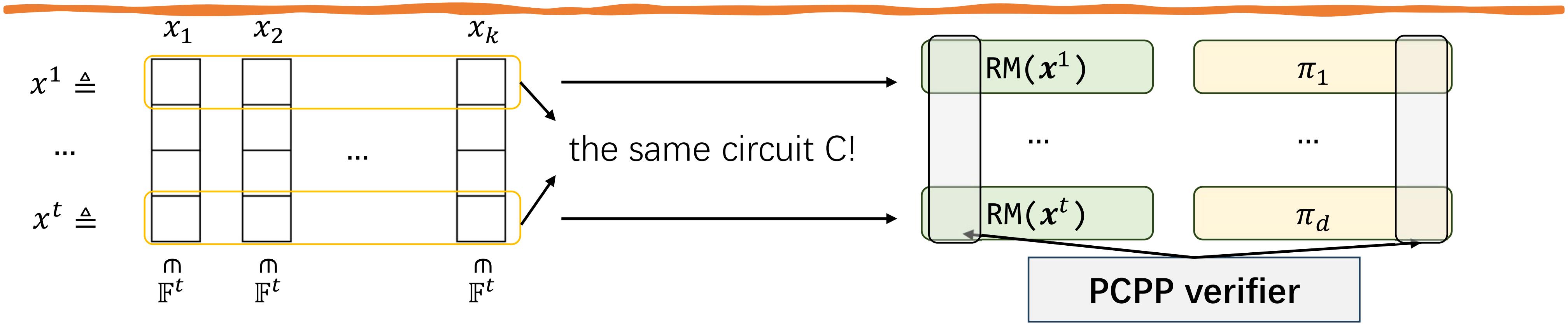
Parallel Encoding

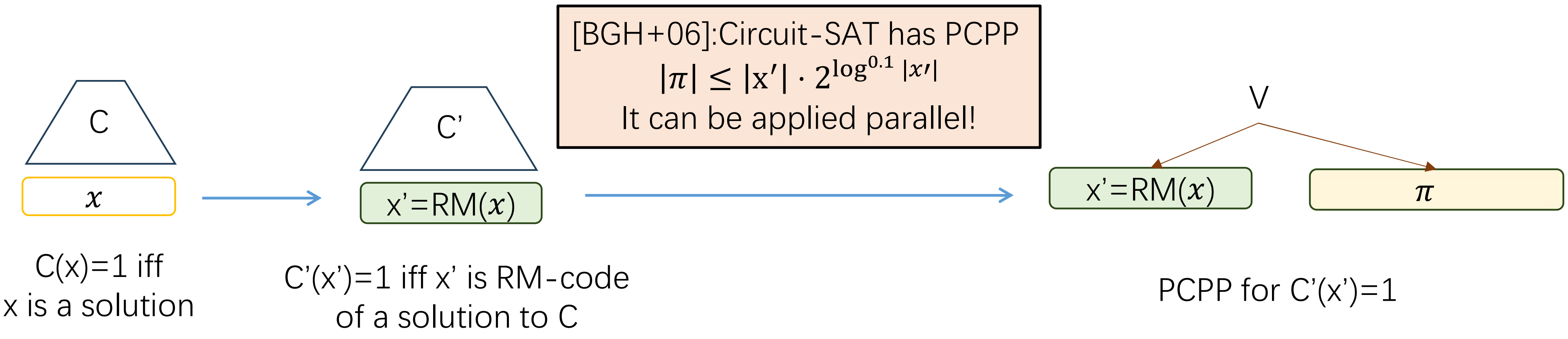
Given a vector-valued CSP with variables $\{x_1, \dots, x_k\}$:



	Code	Parameter blow-up:
[GLRSW24]:	**Hadamard Code:** $\mathbb{F}^k \to \mathbb{F}^{\lvert\mathbb{F}\rvert^k}$	$k \to \lvert\mathbb{F}\rvert^k$
[This work]:	**Reed Muler Code:** $\mathbb{F}^{\lvert\binom{m+d}{d}\rvert} \to \mathbb{F}^{\lvert\mathbb{F}\rvert^m}$ $m = \sqrt{\log k}, d = \sqrt{\log k}\, 2^{O(\sqrt{\log k})}, \binom{m+d}{d} = k, \lvert\mathbb{F}\rvert = O(md)$	$\binom{m+d}{d} = k \to \lvert\mathbb{F}\rvert^m = k 2^{O(\log\log k \sqrt{\log k})}$

PCPP for the Parallel Part





PCPP for the Linear Part

Linear Constraints

$$\overbrace{\begin{aligned} y_1 &= M_1 x_1 \\ y_2 &= M_2 x_2 \\ &\vdots \\ y_k &= M_k x_k \end{aligned}}$$

degree-d $\mathrm{R}M: \mathbb{F}^k \to \mathbb{F}^{|\mathbb{F}|^m}$, where $\mathrm{k} = \binom{m+d}{d}$

$$\mathrm{R}M\,(\boldsymbol{y}_1 \dots, \boldsymbol{y}_k) = (\overbrace{\boldsymbol{y}_1 \dots, \boldsymbol{y}_k}^{\text{Systematic part}}, \mathrm{y}_{k+1} \dots, \mathrm{y}_{|\mathbb{F}|^m})$$
$$\mathrm{R}M\,(\boldsymbol{x}_1 \dots, \boldsymbol{x}_k) = (\boldsymbol{x}_1 \dots, \boldsymbol{x}_k, \mathrm{x}_{k+1} \dots, \mathrm{x}_{|\mathbb{F}|^m})$$
$$\mathrm{R}M\,(\boldsymbol{M}_1 \dots, \boldsymbol{M}_k) = (\boldsymbol{M}_1 \dots, \boldsymbol{M}_k, M_{k+1} \dots, M_{|\mathbb{F}|^m})$$
$$\boldsymbol{z} = (y_1 \dots, y_{|\mathbb{F}|^m}) - (M_1 x_1, \dots, M_{|\mathbb{F}|^m} x_{|\mathbb{F}|^m})$$

PCPP verifier for linear

$\hat{\boldsymbol{x}} = \mathrm{RM}(x_1 \dots, x_k)$ | $\hat{\boldsymbol{y}} = \mathrm{RM}(y_1 \dots, y_k)$ | $\hat{\boldsymbol{z}} = \boldsymbol{z}$ | π

Fact I. $\boldsymbol{z}$ is a codeword of degree-2d RM and $\boldsymbol{z} = (\overbrace{0 \dots, 0}^{\mathrm{k}}, \mathrm{z}_{k+1} \dots, \mathrm{z}_{|\mathbb{F}|^m})$ if the linear constraints are satisfied.

Fact II. If $\hat{\boldsymbol{z}}$ and $\boldsymbol{z}$ are truth-tables of degree-2d polynomials and $\hat{\boldsymbol{z}} \neq \boldsymbol{z}$, then

$$\Pr_{\xi \in \mathbb{F}^m}[\hat{\mathbf{z}}[\xi] \neq \mathrm{z}[\xi]] \geq 1 - O(d/|\mathbb{F}|).$$

Linear Constraints are satisfied if

1) $\hat{\boldsymbol{y}}, \hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}$ are close to some RM codes.
2) the first k entries of the RM code $\hat{\boldsymbol{z}}$ close to are zeros.
3) the RM code $\hat{\boldsymbol{z}}$ close to is $\boldsymbol{z}$.

Same as parallel constraints

Can be checked by randomly picking $\xi \in \mathbb{F}^m$

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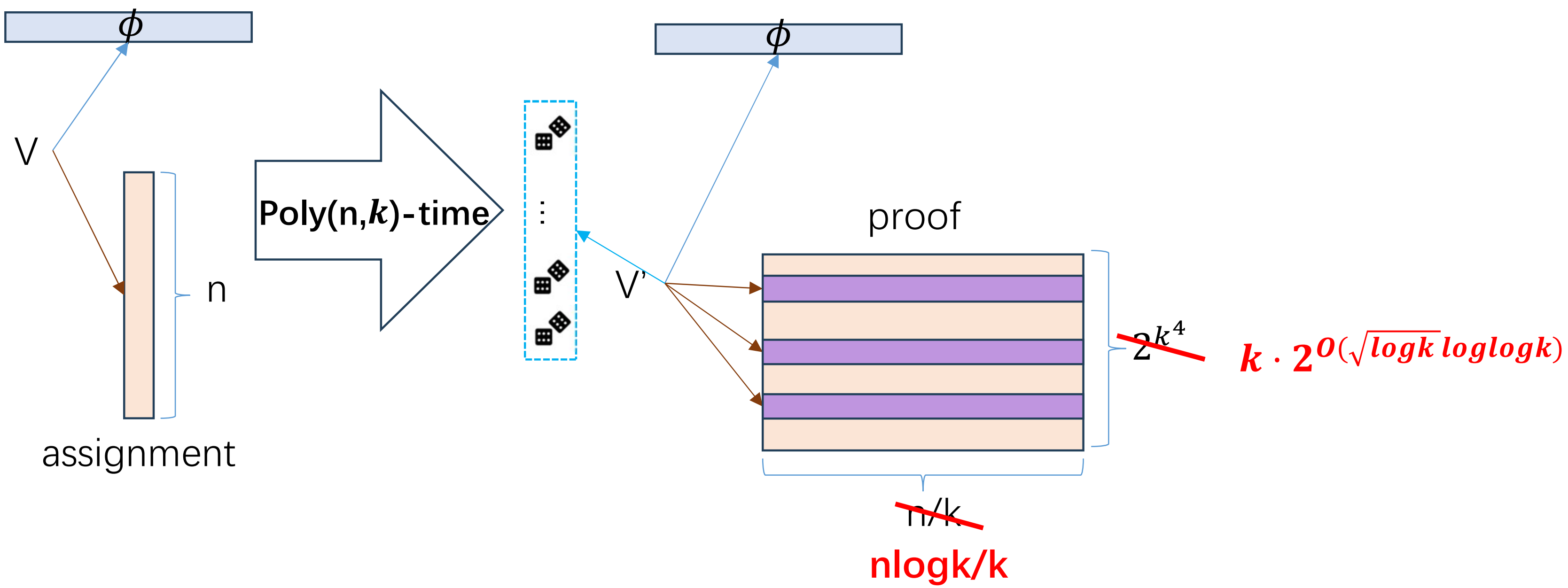

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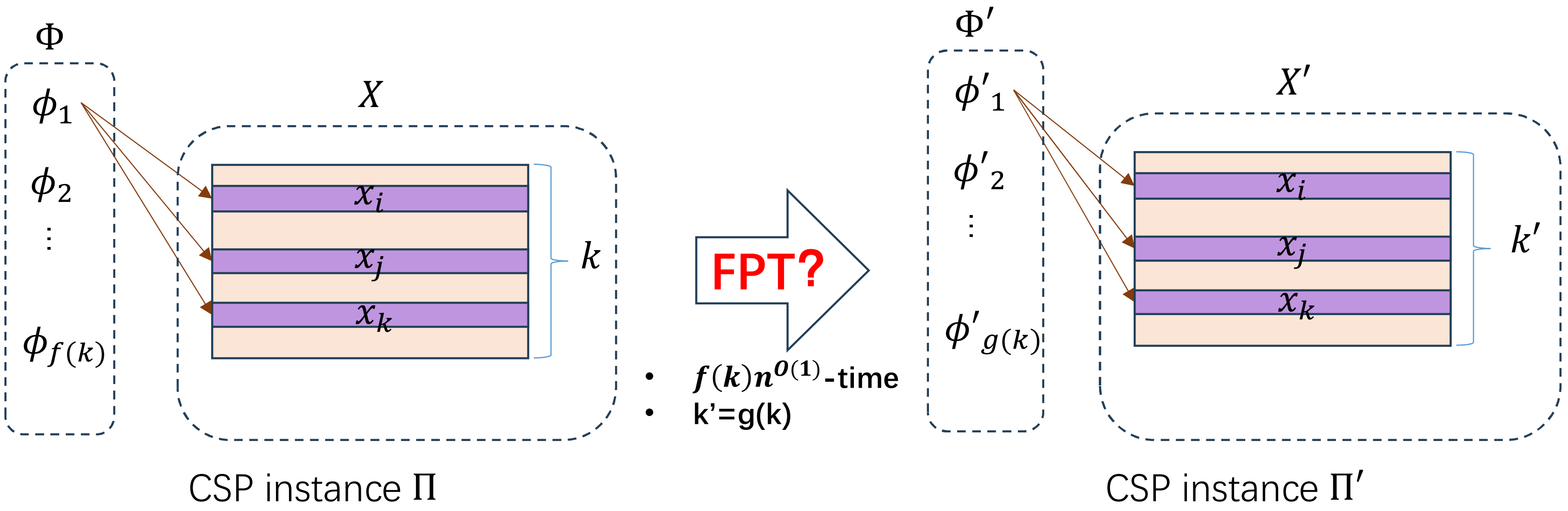
Open Question: W[1]$\neq$FPT $\Rightarrow$ PIH



If val(Π) $= 1$ then val(Π') $= 1$

If val(Π) < 1 then val(Π') $\leq 1 - \epsilon$

Thank you