On the Approximability of Parameterized Minimum Monotone Satisfying Assignment

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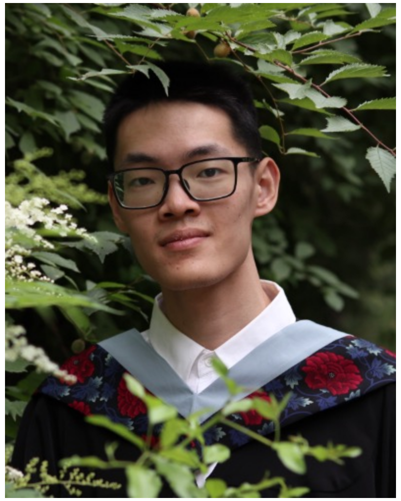


The MMSA Problem

- Minimum Monotone Satisfying Assignment [ABMP'01]
  - `Input:` a monotone Boolean circuit C on n input bits
  - `Output:` $\min|x|_0$, s.t. $x \in \{0,1\}^n, C(x) = 1$

- MMSA$_t$ [DS'04]
  - $\leq t$ levels of alternations, starting with $\land$

NP-Approximability of MMSA

- MMSA_1: Easy
- MMSA_2: Equivalent to Set-Cover
  - $O(\log n)$ approximation [Joh'74, Lov'75, Chv'79]
  - $\big(1 - o(1)\big) \ln n$ inapproximability [DS'14]
- MMSA_3:
  - $n^{O(1)}$ approximation [CMV'23]
  - $2^{\log^{1-o(1)} n}$ inapproximability [DS'04]
- $\text{MMSA}_3 \leq$ Label-Cover $\leq \text{MMSA}_4$ [DS'04]

- Semantic MMSA:
  - $n^{1/5-\varepsilon}$ hardness [Uma'99]

Parameterized Approximability of MMSA

- k-MMSA
  - k: optimum (a.k.a. the minimum Hamming weight)
  - Is there an $f(k)n^{O(1)}$ time (approximation) algorithm?
- Unbounded Alternations:
  - W[P]-complete (circuit version) / W[SAT]-complete (formula version)
- k-MMSA$_2$: Equivalent to k-SetCover
  - $(\log n)^{1/\mathrm{poly}(k)}$ inapprox. under ETH [Lin'19, KLM'19]
  - $(\log n)^{o(1)}$ inapprox. under W[1]≠FPT [Lin'19, KLM'19]
  - $\omega(1)$ inapprox. under W[2]≠FPT [LRSW'23]
- k-MMSA$_4$: Harder than k-MinLabel
  - $n^{1/\mathrm{poly}(k)}$ inapprox. under ETH [KLM'19, KN'21]
  - $n^{o(1)}$ inapprox. under W[1]≠FPT [KLM'19, KN'21]
  - any $f(k)$ inapprox. under W[2]≠FPT [Mar'13]

Q1: Hardness of k-MMSA$_3$?

Q2: Close the gap of approx./inapprox. for k-SetCover?

Our Results

- An FPT-time $O(2^k \log n)$-approximation algorithm for k-MMSA$_3$
- A reduction from (k, h)-gap k-MMSA$_3$ to $(2^k, h/k)$-gap k-MMSA$_2$
- An improved inapproximability for k-MMSA$_4$

The $O(2^k \log n)$-approx. for k-MMSA$_3$

- $C(x) = \bigwedge_{i=1}^{n} \bigvee_{j=1}^{n} \bigwedge_{t \in N(i,j)} x_t$

- Each time, pick the set $N(i,j)$ maximizing the number of newly covered i-indices
- Let x^* be the optimal solution which has weight k
  - x^* “covers” every i-index
  - $\mathrm{supp}(x)$ has $\leq 2^k$ subsets
  - the best subset covers at least a $1/2^k$ fraction
- Each $N(i,j)$ has size $\leq k$, giving $O\left(k 2^k \log n\right)$ in total

(k,h)-gap k-MMSA$_3$ $\rightarrow$ $(2^k, h/k)$-gap k-MMSA$_2$

- $C(x) = \bigwedge_{i=1}^{n} \bigvee_{j=1}^{n} \bigwedge_{t \in N(i,j)} x_t$

- Build a variable for each set $N(i,j)$:
- $F(z) = \bigwedge_{i=1}^{n} \bigvee_{j=1}^{n} z_{N(i,j)}$

- Completeness:
  - suppose $\exists |x|_0 \leq k, C(x) = 1$, set every z_S where $S \subseteq \mathrm{supp}(x)$ to 1. $|z|_0 \leq 2^k$.
- Soundness:
  - take the union of the selected $N(i,j)$'s, where each $N(i,j)$ has size $\leq k$

k-MMSA$_t \rightarrow$ gap k-MMSA$_{t+2}$

- Revisiting [Mar'13]'s reduction
- `Ingredient:` a threshold graph $G = (A, B_1 \cup \cdots \cup B_m, E)$. Informally:
  - any k vertices in A have a common neighbor in every B_i,
  - but any $(k+1)$ vertices in A "disagree" on at least one B_i

- $C'(x) = \bigwedge_{i=1}^{m} \bigvee_{b \in B_i} C(x \wedge 1_{N(b)})$

- We instantiate [Mar'13]'s reduction with a better threshold graph, yielding a better hardness for k-MMSA$_4$.

Open Questions

- A better inapproximability of k-MMSA$_2$ (a.k.a. k-SetCover) through k-MMSA$_3$?
  - $(\log n)^{\Omega(1)}$ inapprox. for k-MMSA$_3$ under W[1]≠FPT → same for k-MMSA$_2$
  - $\omega(2^k)$ inapprox. for k-MMSA$_3$ under W[2]≠FPT → super-constant for k-MMSA$_2$

- A $(\log n)^{0.99}$ FPT approximation algorithm for k-MMSA$_2$?
- An $n^{1/100}$-inapproximability of k-MMSA$_{100}$?

Thanks!