Constant Approximating Parameterized k-SetCover is W[2]-hard

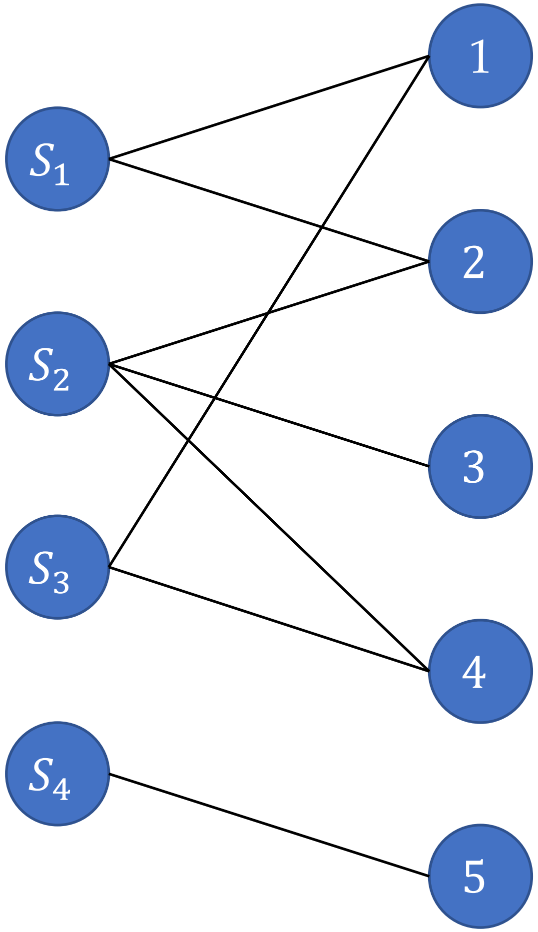
Bingkai Lin, *Xuandi Ren*, Yican Sun, Xiuhan Wang

Jan, 2023

SetCover Problem

Input: $(\mathcal{S}, U)$, where $\mathcal{S}$ is a collection of n sets $S_1, \dots, S_n$ over the universe U.
Output: find the smallest number of sets in $\mathcal{S}$, whose union is U.

- Equivalent view:
  - Given a bipartite graph $G = (\mathcal{S} \cup U, E = \{(S_i, u) | u \in S_i\})$, find the smallest number of left vertices, whose neighbors' union is U.
- Example:
  - $U = \{1,2,3,4,5\}, \mathcal{S} = \{S_1 = \{1,2\}, S_2 = \{2,3,4\}, S_3 = \{1,4\}, S_4 = \{5\}\}$
  - Answer: 3 (S_1, S_2, S_4).



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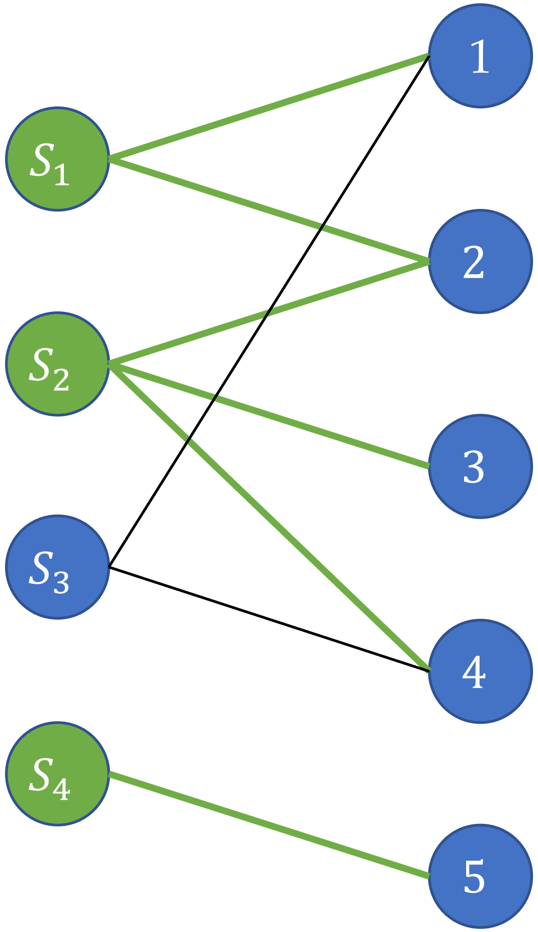
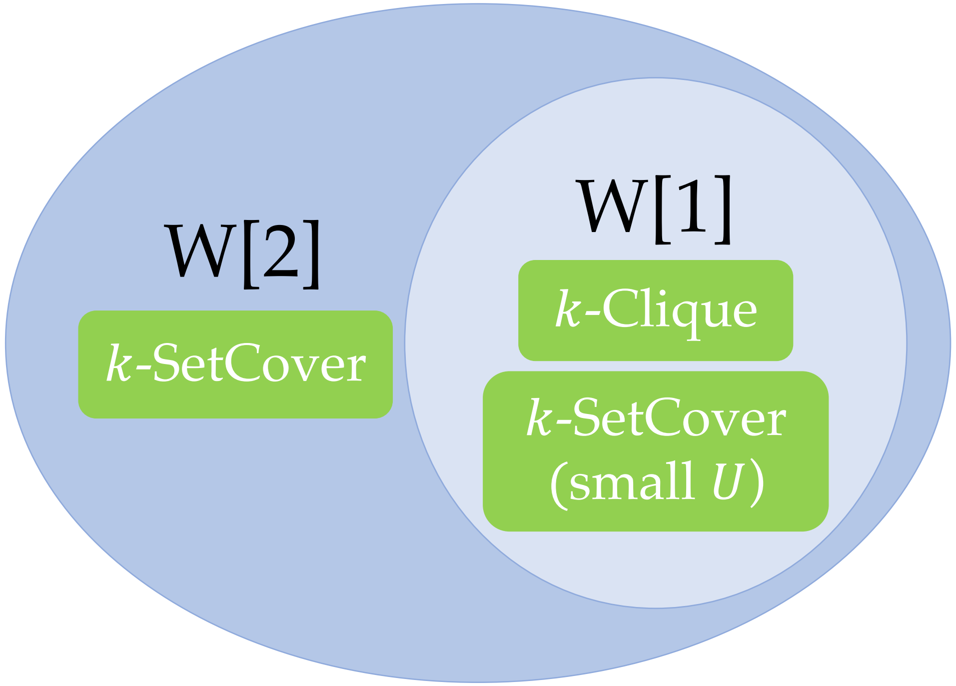
  - Answer: 3 (S_1, S_2, S_4).
- NP-complete [Kar72]



Parameterized SetCover Problem

> ***k*-SetCover**
> **Input:** $(\mathcal{S}, U, k)$, where $\mathcal{S}$ is a collection of n sets $S_1, \ldots, S_n$ over the universe U.
> **Output:** find the smallest number of sets in $\mathcal{S}$, whose union is U. It's guaranteed that the optimum is k.

- W[2]-complete [DF95]
  - W[2]≠FPT ⇒ no algorithm in $f(k)n^{O(1)}$ time
- W[1]-complete if $|U| = \Theta(k^{O(1)} \log n)$.
  - [KLM19,Lin19] + **This work**
- Naïve algorithm:
  - enumerating all k-tuple of sets in $n^{O(k)}$ time
  - dynamic programming on subsets of U in $2^{O(|U|)}$ time



Approximation of SetCover Problem

- "The set cover problem plays the same role in approximation algorithms that the maximum matching problem played in exact algorithms - as a problem whose study led to the development of fundamental techniques for the entire field."

  - *Approximation Algorithms,* by Vijay V. Vazirani

Approximation of SetCover Problem

- A simple greedy algorithm reaches $(\ln n - \ln\ln n + \Theta(1))$ approximation ratio [Sla97].

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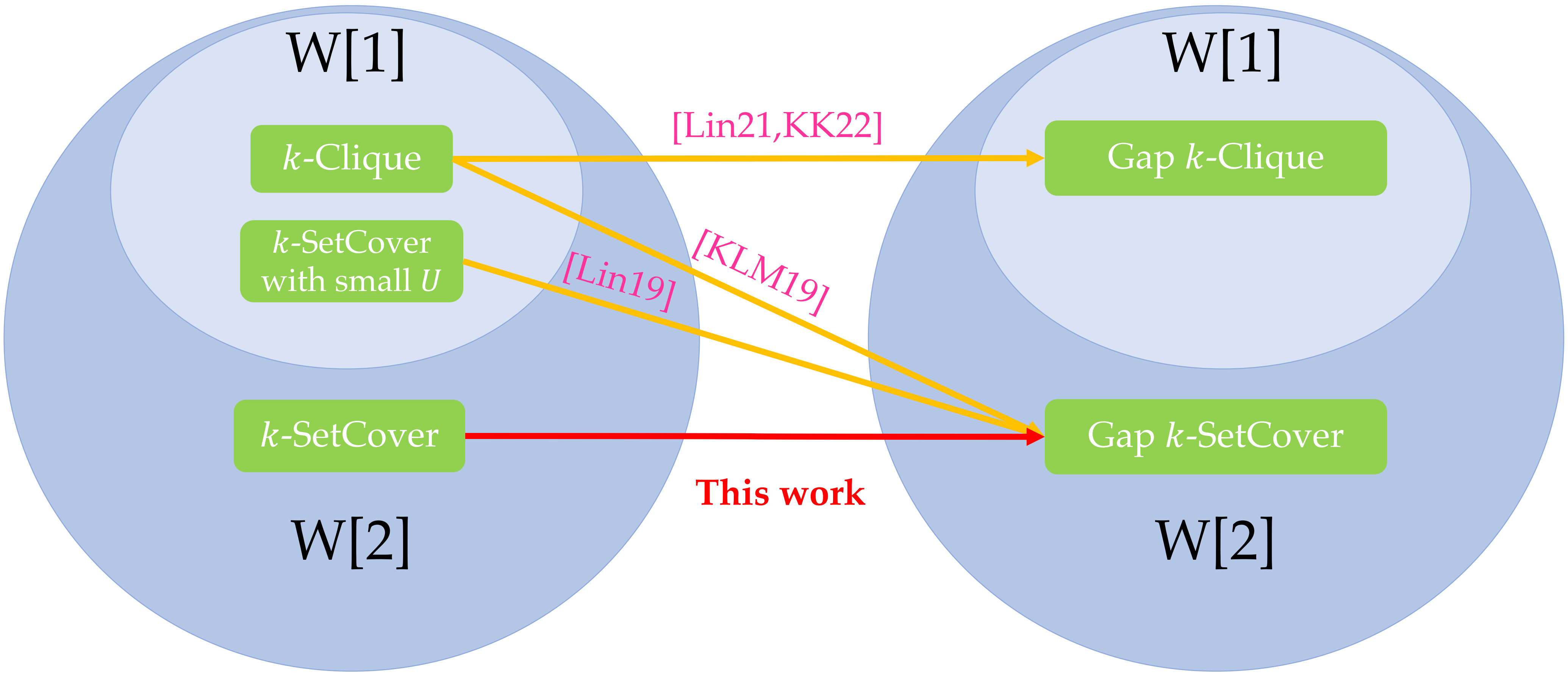
W[2]-hardness of Approx. k-SetCover



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Problem	Assumption	Hardness of Approx. Ratio	Running Time Bound	Reference
SetCover	NP≠P	$(1-\varepsilon)\ln n$	$n^{O(1)}$	[DS14]
Parameterized SetCover	SETH	$\left(\frac{\log n}{\log\log n}\right)^{1/k}$	$f(k)n^{k-\varepsilon}$	[KLM19, Lin19]
	ETH		$f(k)n^{o(k)}$	
	k-SUM Hypothesis		$f(k)n^{\lceil\frac{k}{2}\rceil-\varepsilon}$	
	W[1]≠FPT	$(\log n)^{\varepsilon(k)}$	$f(k)n^{O(1)}$	
	W[2]≠FPT	?	?	

Approx. SetCover with Small OPT Size

Problem	Assumption	Hardness of Approx. Ratio	OPT Size	Technique	Reference
SetCover	NP≠P	$(1-\varepsilon)\ln n$	$\Omega(n)$	(Complex) PCP	[DS14]
	W[1]≠FPT	$o\left(\frac{\log n}{\log\log n}\right)$	$O(\text{poly}\log n)$	New TGC	**This work**
	W[1]≠FPT	$(\log n)^{\varepsilon(k)}$	k	TGC / Distributed PCP	[KLM19, Lin19]

Our Technique

- Threshold Graph Composition [Lin18, CL19, Lin19, BBE+21]
    - exploited new properties of threshold graphs
    - used the construction from error-correcting codes [KN21]
    - discovered new composition scheme

Constant Approximating k-SetCover is W[2]-hard!

k-SetCover Instance + Threshold Graphs = Constant Gap k'-SetCover Instance

Threshold Graph in [Lin19]

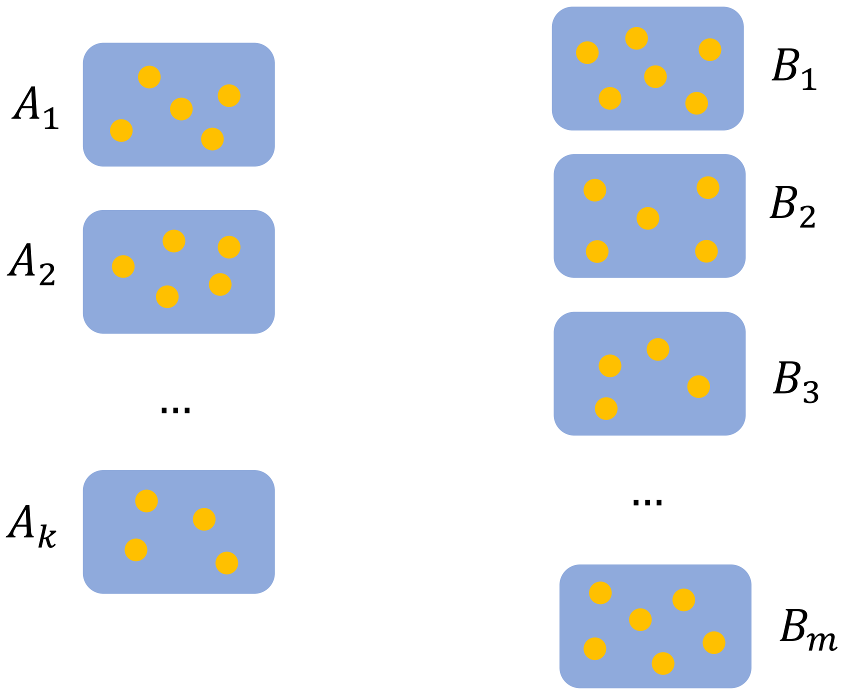
- Threshold Graph G_T: a bipartite graph $(A \cup B, E)$ with $A = A_1 \cup \cdots \cup A_k$ and $B = B_1 \cup \cdots \cup B_m$, satisfying

Completeness:

- $\forall a_1 \in A_1, \ldots, a_k \in A_k$ and $i \in [m]$, $a_1, \ldots, a_k$ have a common neighbor in B_i

Soundness:

- $\forall X \subseteq A$ if $\forall i \in [m], \exists b_i \in B_i$ such that b_i has $k+1$ neighbors in X, then $|X| > h$



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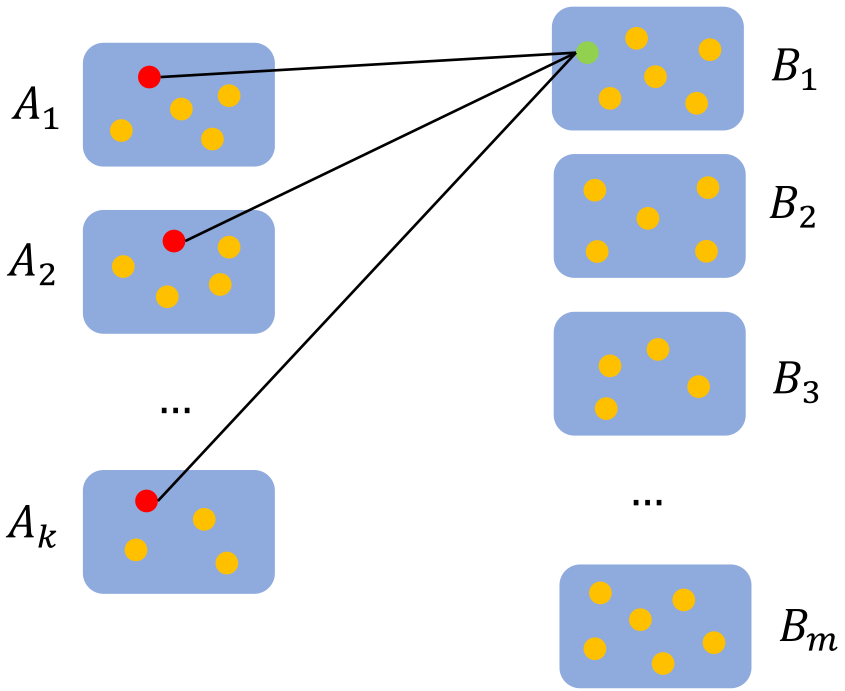

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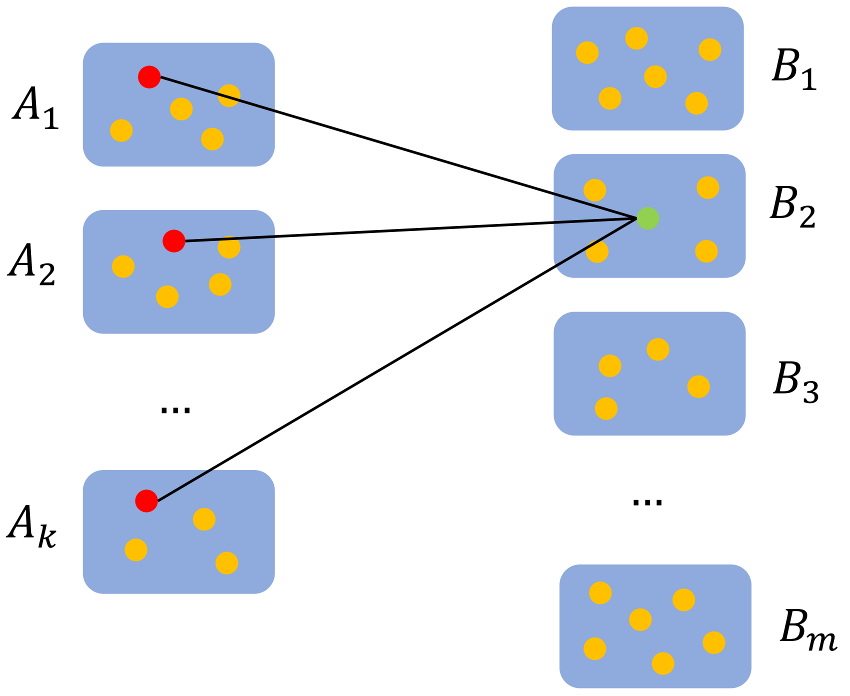

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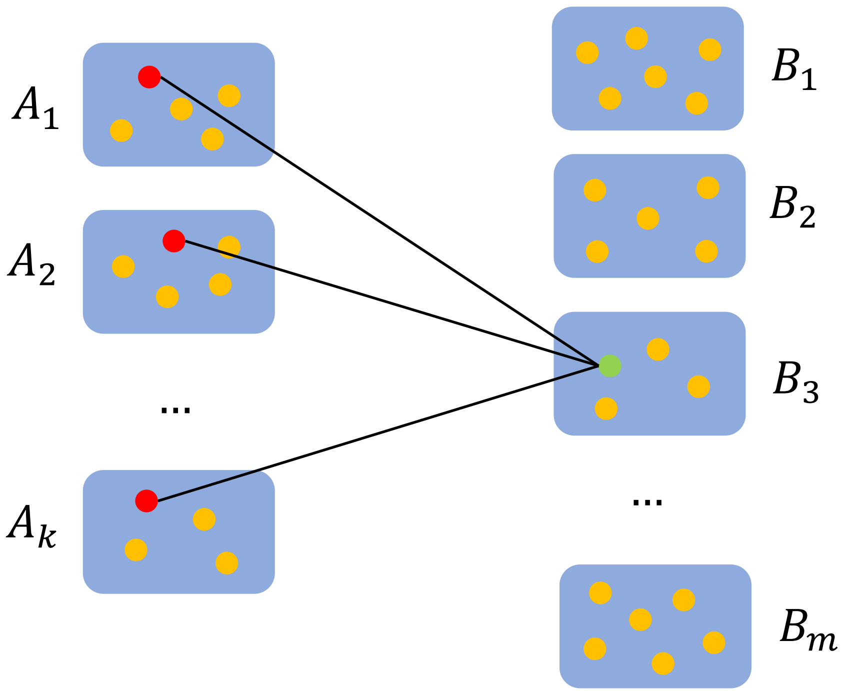

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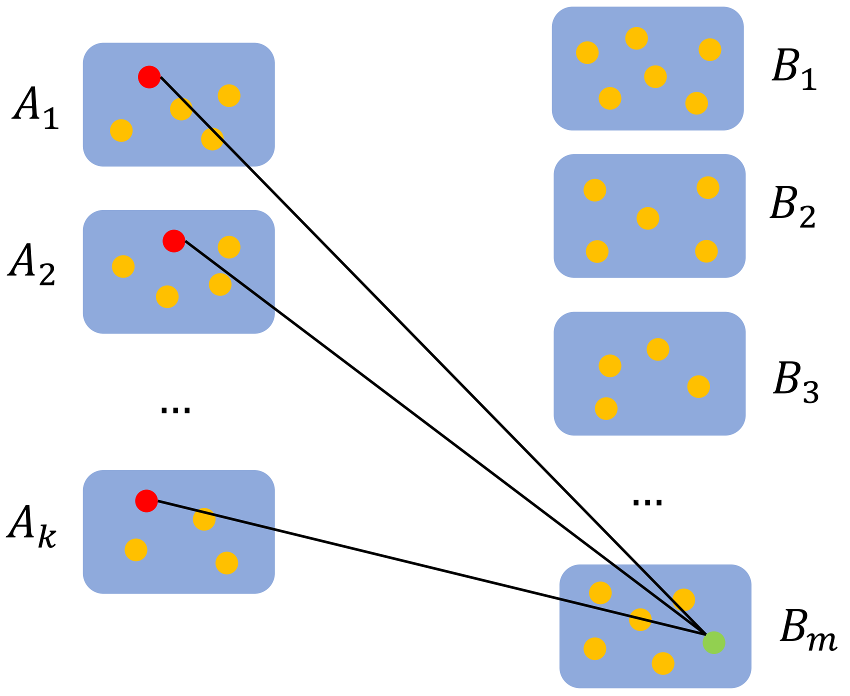

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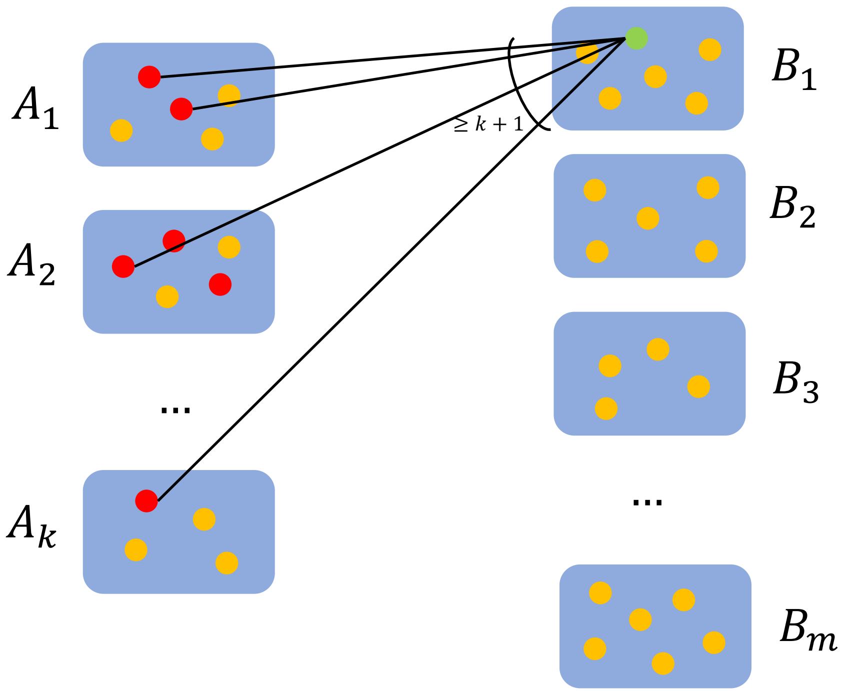

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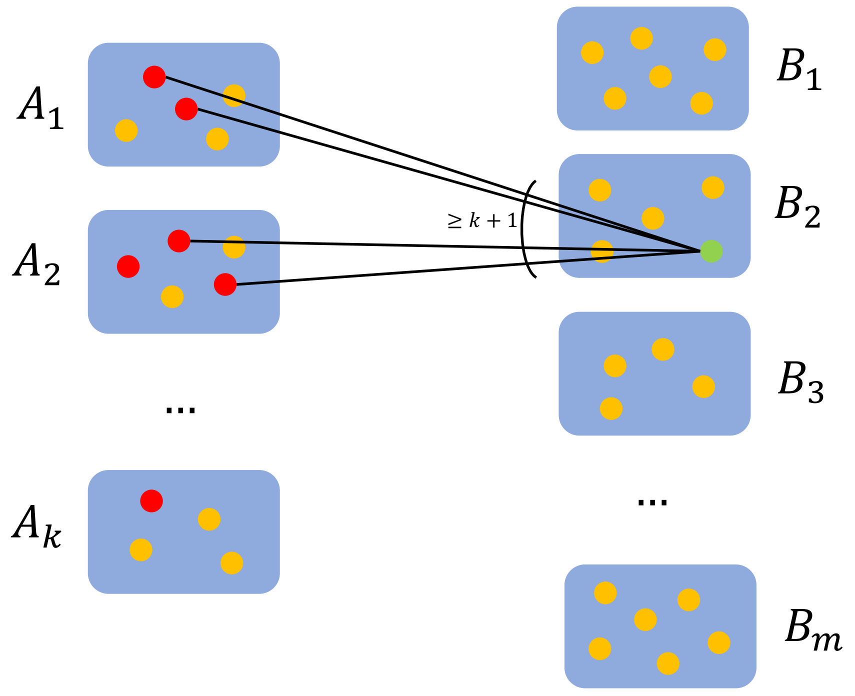

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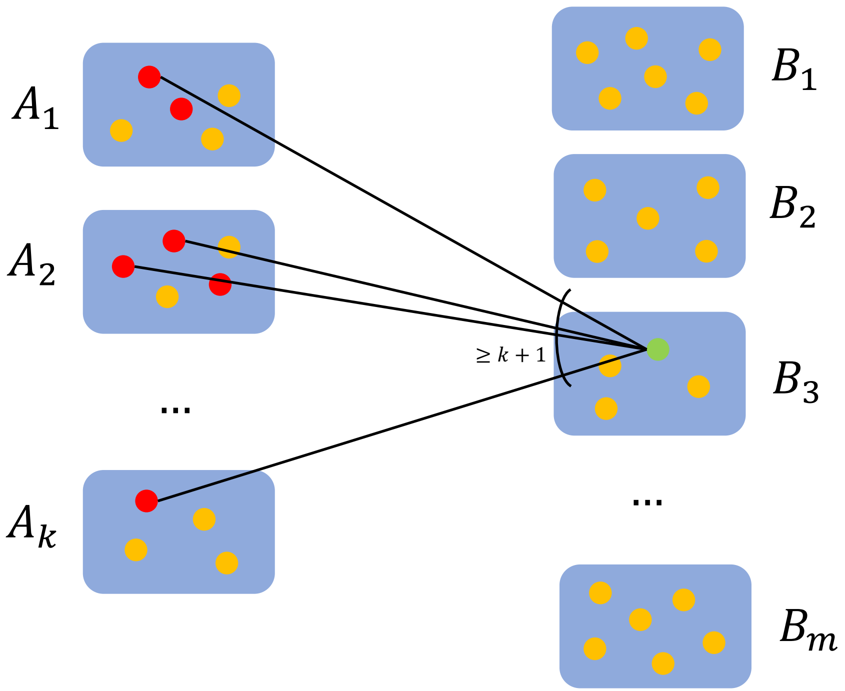

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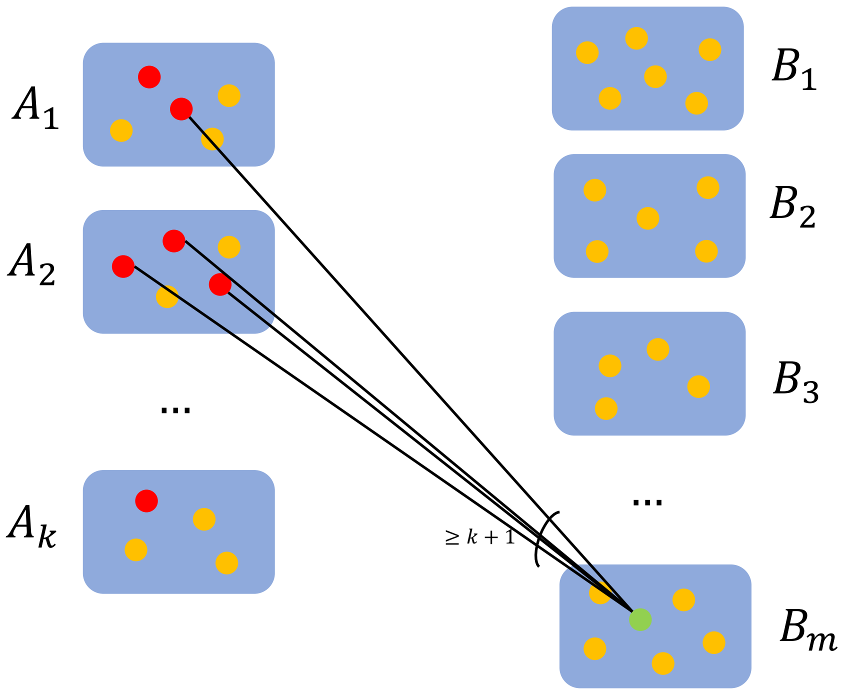

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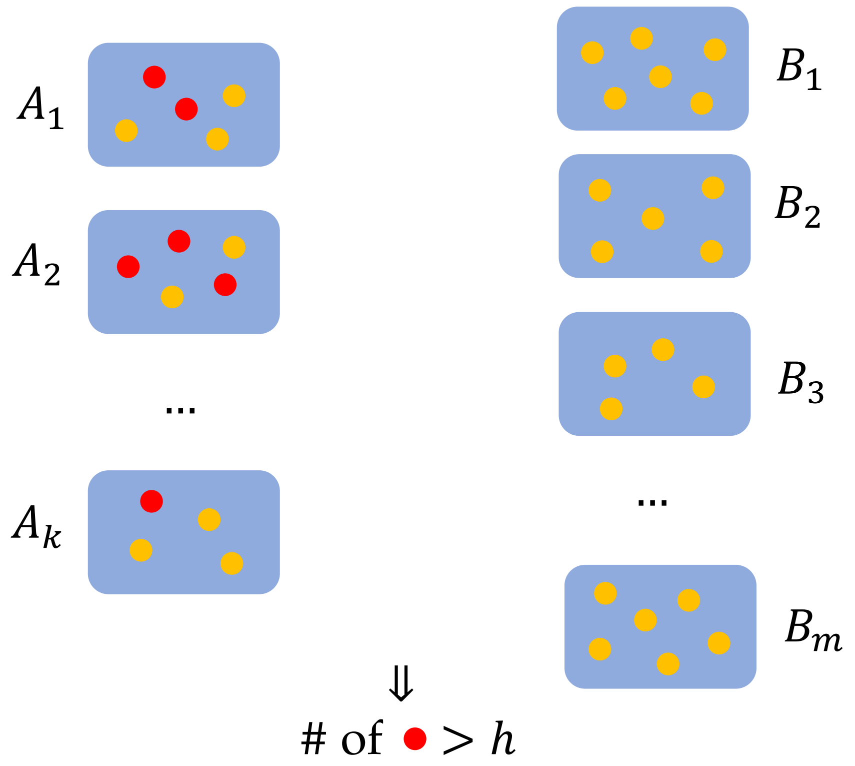

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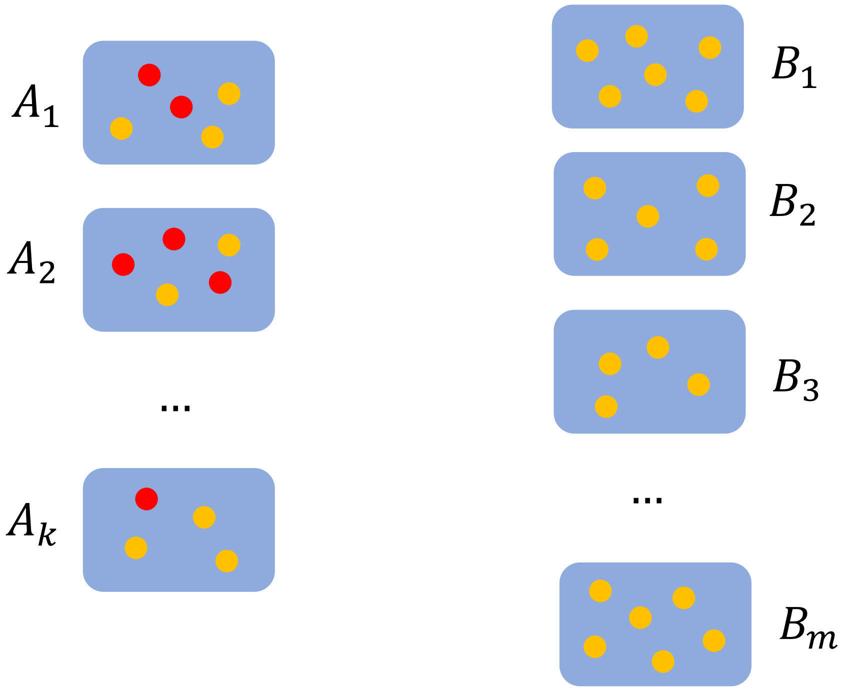

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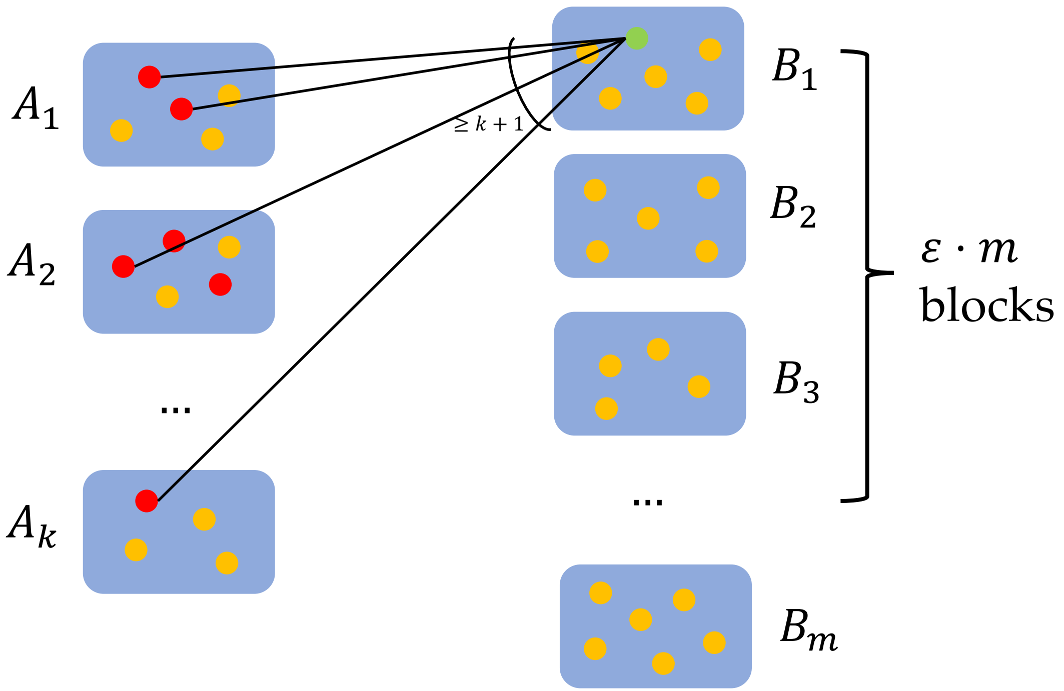

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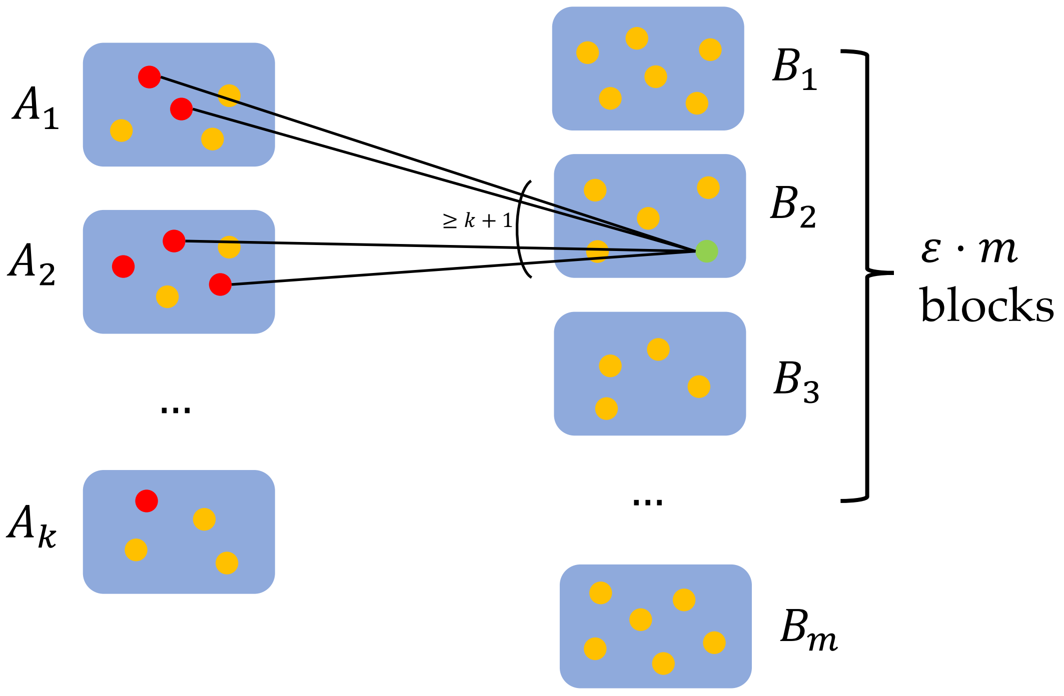

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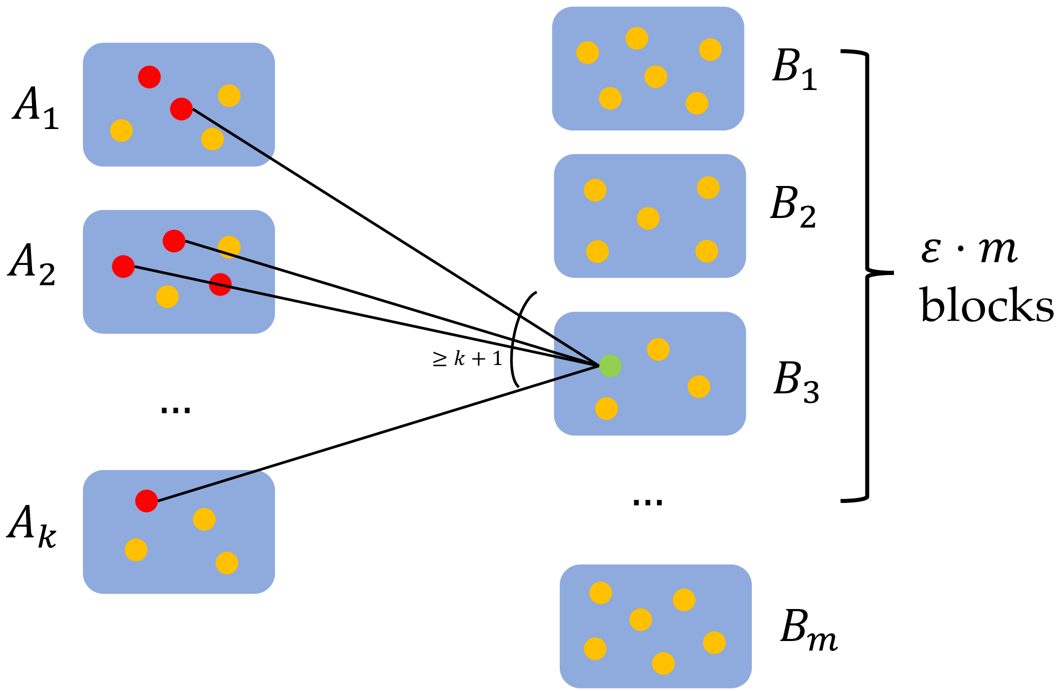

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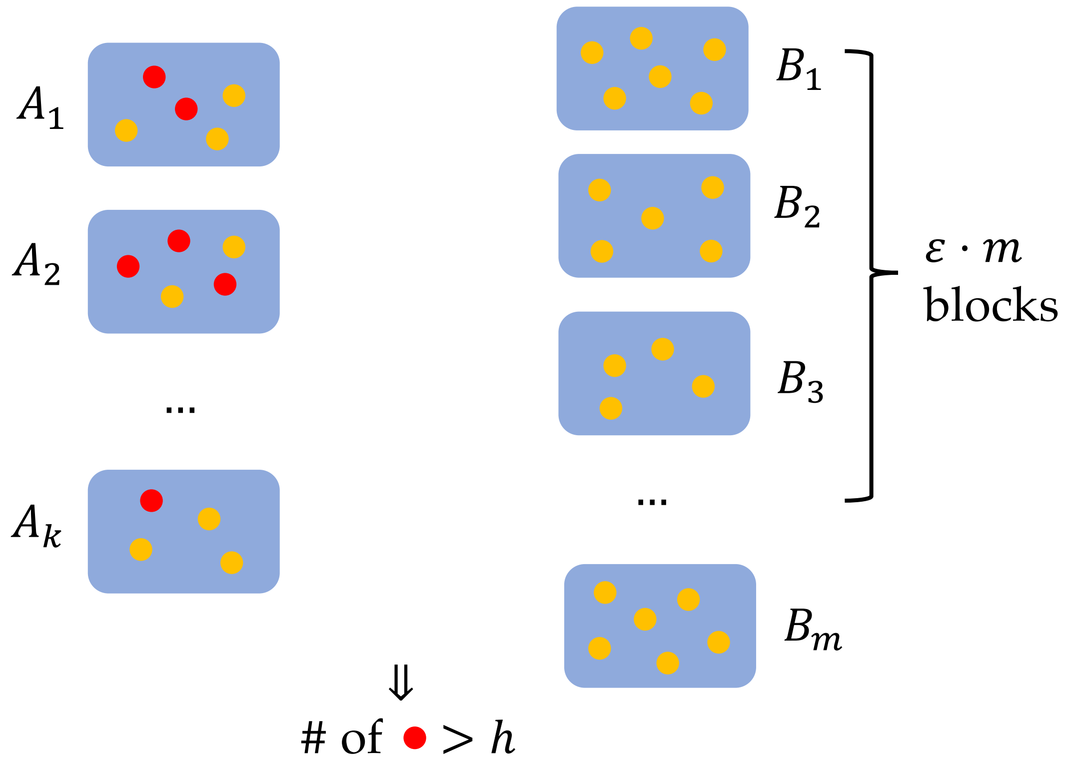

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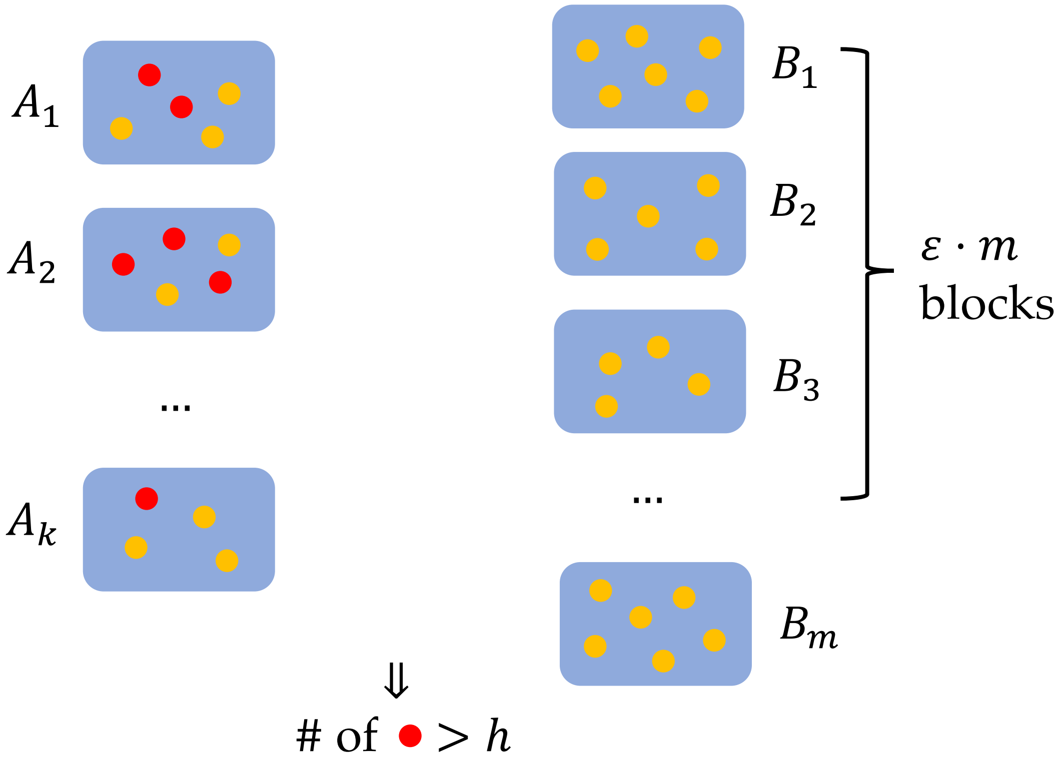

- Such threshold graphs can be constructed via error correcting codes with benign distance [KN21]!

Threshold Graph Composition in [Lin19]

k-SetCover Instance $\Gamma = (\mathcal{S}, U)$

YES Instance:

- $\exists S_1, \ldots, S_k \in \mathcal{S}$ which can cover U

NO Instance:

- any covering of U has size $\geq k+1$

Threshold Graph $G_T = (A \cup B, E)$

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c-gap k-SetCover Instance $\Gamma' = (\mathcal{S}', U')$, where $|U'| = |U|^{|B_i|}$

YES Instance:

- $\exists S_1, \ldots, S_k \in \mathcal{S}'$ which can cover U'

NO Instance:

- any covering of U has size $\geq h$

Threshold Graph Composition

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c-gap k'-SetCover Instance $\Gamma' = (\mathcal{S}', U')$, where $|U'| = (|U||B_i|)^{O(c)}$

YES Instance:

- $\exists S_1, \dots, S_{k'} \in \mathcal{S}'$ which can cover U'

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Threshold Graph Composition

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Our Reduction

- Treat every A_i as a copy of $\mathcal{S}$.
- The new sets $\mathcal{S}' = A \cup B$.
- The new universe U' has m parts $U'_1, \dots, U'_m$.

Desired Property

For any $X \subseteq A$ and $Y \subseteq B$ can cover U'_i iff

1. either $\exists b_i \in Y \cap B_i$, s.t. $X \cap \mathcal{N}_{G_T}(b_i)$ cover U,
2. or $|Y \cap B_i| \geq c+1$.

Analysis of the YES Case

YES Instance:
- $\exists S_1, \ldots, S_{k'} \in \mathcal{S}'$ which can cover U'

k-SetCover Instance $\Gamma = (\mathcal{S}, U)$

YES Instance:
- $\exists S_1, \ldots, S_k \in \mathcal{S}$ which can cover U

NO Instance:
- any covering of U has size $\geq k + 1$

Threshold Graph $G_T = (A \cup B, E)$

Completeness:
- $\forall a_1 \in A_1, \ldots, a_k \in A_k$ and $i \in [m]$, $a_1, \ldots, a_k$ have a common neighbor in B_i

Soundness:
- $\forall X \subseteq A$ if for ε fraction of $i \in [m]$, $\exists b_i \in B_i$ such that b_i has $k + 1$ neighbors in X, then $|X| > ck'$

Our Reduction
- Treat every A_i as a copy of $\mathcal{S}$.
- The new sets $\mathcal{S}' = A \cup B$.
- The new universe U' has m parts $U'_1, \ldots, U'_m$.

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2 or $|Y \cap B_i| \geq c + 1$.

In the YES case, it suffices to pick $S_1, \ldots, S_k$ and their common neighbors in every B_i.

In the NO case, one of the following holds:
- (# of 1 $\geq \varepsilon m$) $\Rightarrow$ $|X| > ck'$.
- (# of 2 $\geq (1 - \varepsilon)m$) $\Rightarrow |Y| \geq (c + 1)(1 - \varepsilon)m$,

Analysis of the NO case

NO Instance:
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Summary

Theorem 1. Assuming W[2]≠FPT, there is no FPT algorithm which can approximate k-SetCover within any constant ratio.

Theorem 2. Assuming W[1]≠FPT, there is no polynomial time algorithm which can approximate k-SetCover within $o\left(\frac{\log n}{\log\log n}\right)$ ratio, even if k is as small as $O\left(\frac{\log n}{\log\log n}\right)^3$.

Open Questions

Open Question 1. *Is it W[2]-hard to approximate k-SetCover within $\omega(1)$ ratio?*

- Our reduction has running time $\Omega(|U|^c)$, thus can not get super-constant inapproximability.

Open Question 2. *Is there any FPT algorithm which can approximate k-SetCover with approximation ratio $o(\log n)$?*

- Current best lower bound is $(\log n)^{\varepsilon(k)}$ for any function $\varepsilon(k) = o(1)$.

Q&A

- Thanks!