On Lower Bounds of Approximating Parameterized k-Clique

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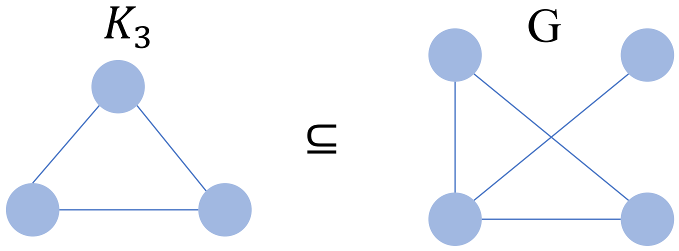
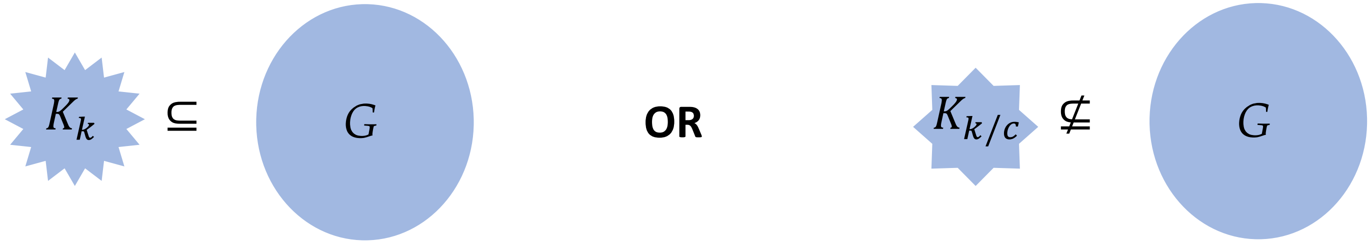
k-Clique Problem

- Given a graph G with a positive integer k, decide if G contains K_k—a complete subgraph of size k



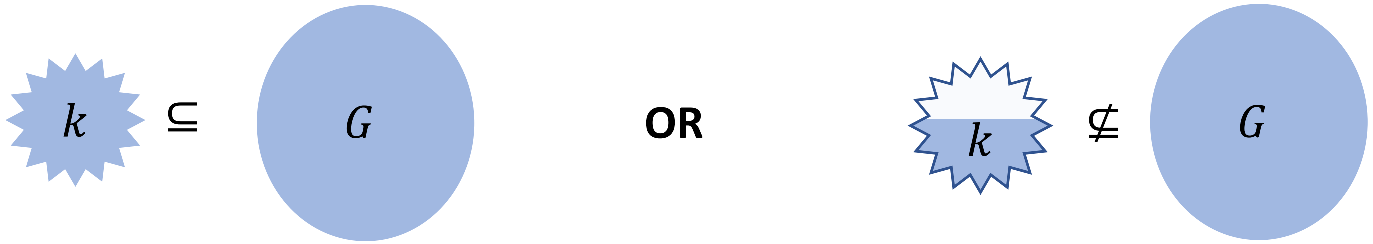
- In the **c-approximation (c-gap)** version, distinguish between:
  - G has k-clique
  - G has no k/c-clique



Densest k-Subgraph:
Given a graph G and a positive integer k, choose k vertices such that they induce as many edges as possible

***c*-approximation** version: distinguish

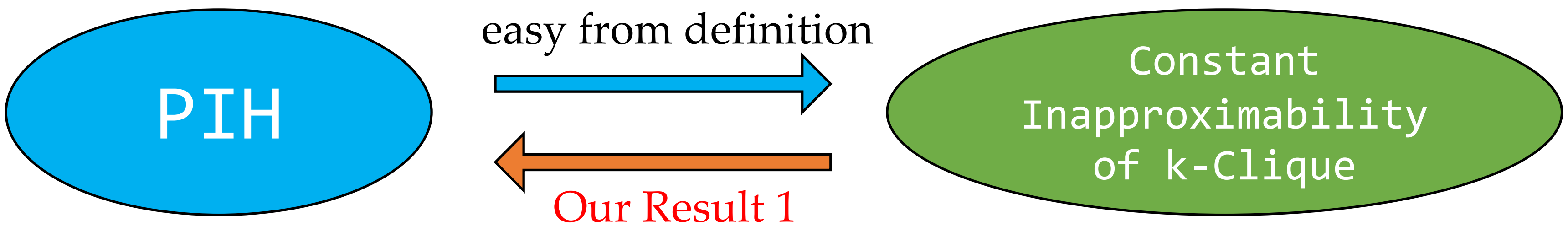
- G has k-clique
- Any k vertices induce at most $\binom{k}{2}/c$ edges



Parameterized Inapproximability Hypothesis [Lokshtanov-Ramanujan-Saurabh-Zehavi'20]
There is no constant approximation **FPT**($f(k) \cdot n^{O(1)}$**-time**) algorithm for **Densest k-Subgraph**

Parameterized PCP-theorem!

PIH and Gap k-Clique

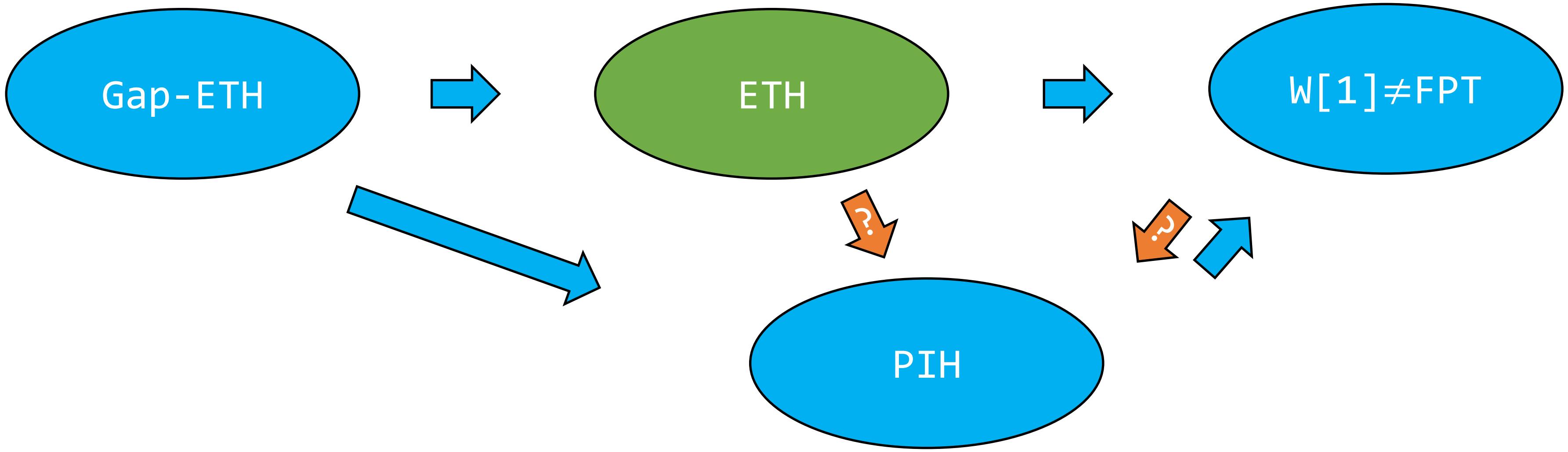


Open problem [Feldmann-Karthik-Lee-Manurangsi'20]:
Does **PIH** hold if we assume that constant Gap k-Clique has no **FPT** algorithm?

- **Result 1:** An $\boldsymbol{f(k) \cdot n^{\Omega\left(\frac{k}{\log k}\right)}}$-time lower bound for constant approximating k-Clique would imply **PIH**.
  - A new potential way to prove **PIH**.

Previous Works for Gap-k-Clique

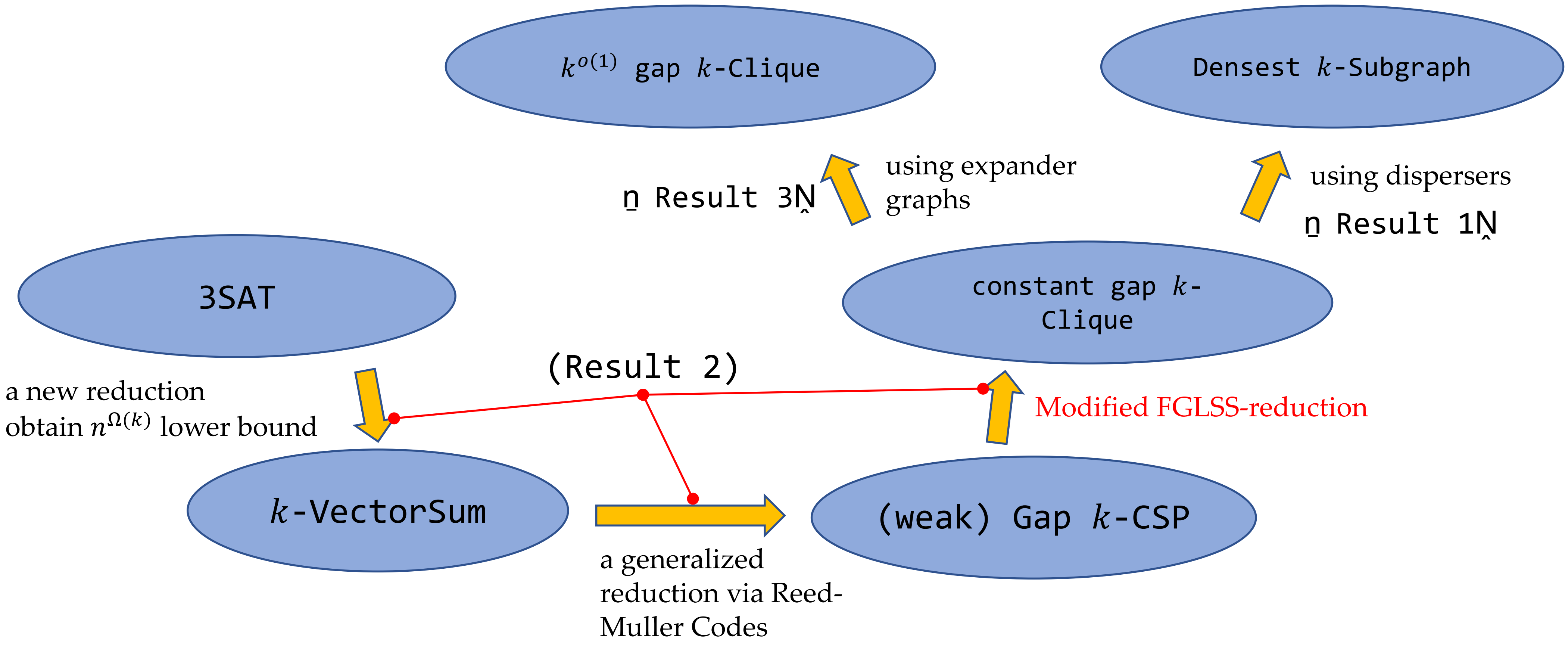
Complexity Assumption	Inapproximability Ratio	Time Lower Bound	Reference
Gap-ETH	$\rho = o(k)$	$f(k) \cdot n^{\Omega\left(\frac{k}{\rho}\right)}$	[CCK+17]
ETH	Any constant	$f(k) \cdot n^{\Omega(\sqrt[6]{\log k})}$	[Lin21]
W[1]≠FPT	Any constant	$f(k) \cdot n^{\Omega(1)}$	[Lin21]
	$\rho = k^{o(1)}$	$f(k) \cdot n^{\Omega(1)}$	[KK22]
PIH	Any constant	$f(k) \cdot n^{\Omega(1)}$	[LRSZ20]



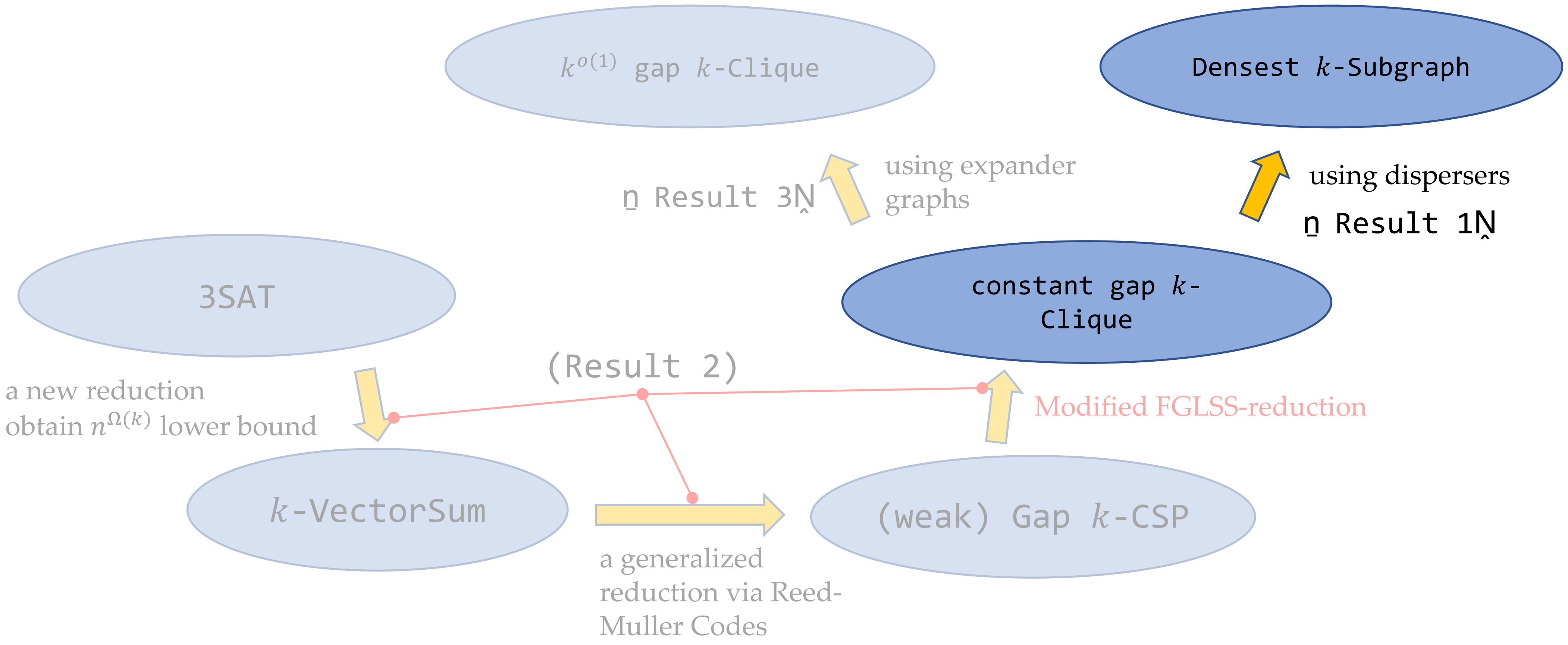
Our Results

Complexity Assumption	Inapproximability Ratio	Time Lower Bound	Reference
Gap-ETH	$\rho = o(k)$	$f(k) \cdot n^{\Omega\left(\frac{k}{\rho}\right)}$	[CCK+17]
ETH	Any constant	$f(k) \cdot n^{\Omega\left(\sqrt[6]{\log k}\right)}$	[Lin21]
	Any constant	$f(k) \cdot n^{\Omega(\log k)}$	Result 2 of This Work
	$\rho = k^{o(1)}$	$f(k) \cdot n^{\Omega(1)}$	Result 3 of This Work
W[1]≠FPT	Any constant	$f(k) \cdot n^{\Omega(1)}$	[Lin21]
	$\rho = k^{o(1)}$	$f(k) \cdot n^{\Omega(1)}$	[KK22]
PIH	Any constant	$f(k) \cdot n^{\Omega(1)}$	[LRSZ20]

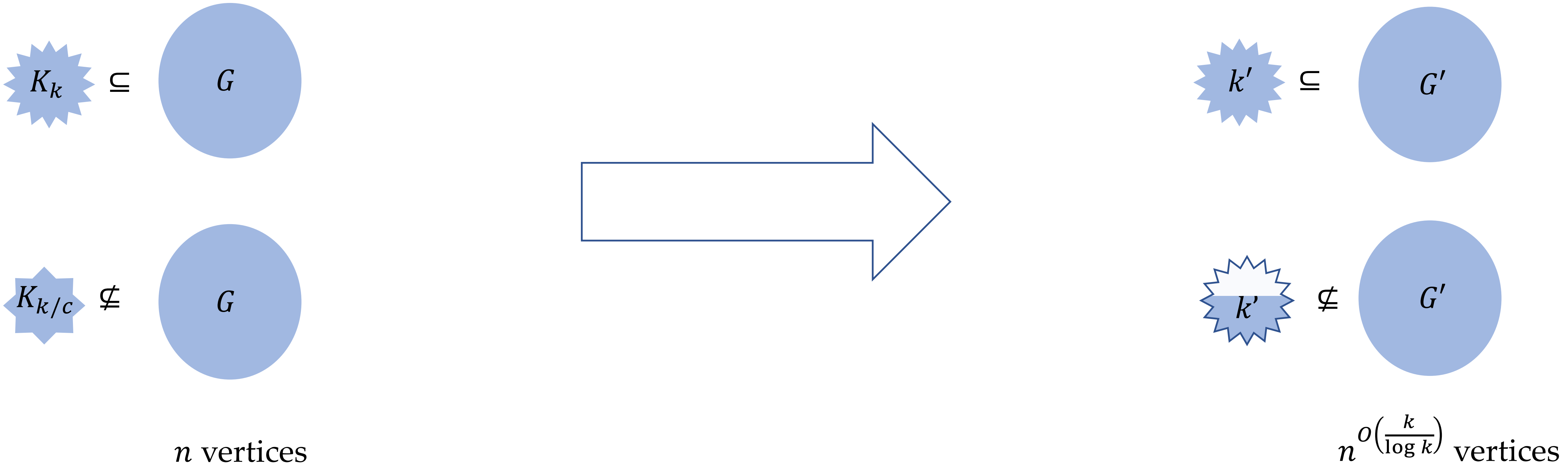
Our Techniques



Our Techniques



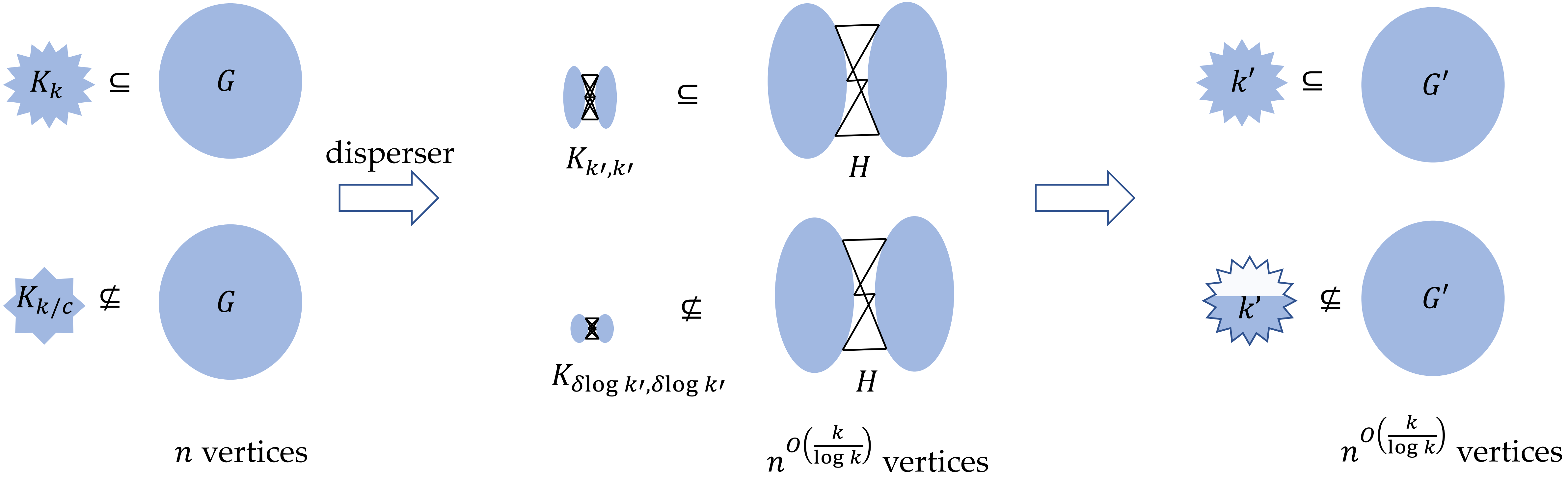
From Gap k-Clique to Gap-Densest-k-Subgraph



Result 1

An $\boldsymbol{f(k) \cdot n^{\Omega\left(\frac{k}{\log k}\right)}}$-time lower bound for constant approximating k-Clique would imply **PIH**.

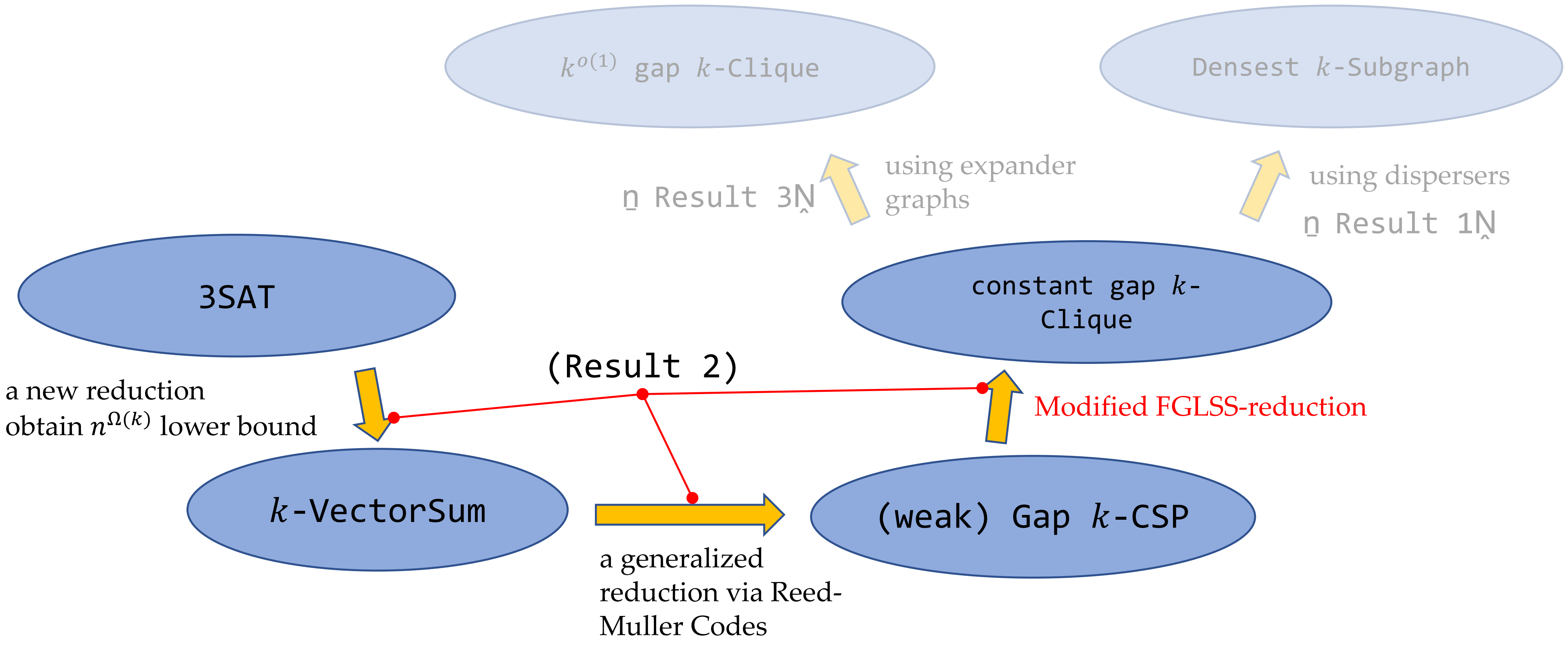
From Gap k-Clique to Gap-Densest-k-Subgraph



Kővári–Sós–Turán theorem
Any $O(k)$-vertex graph without $K_{\delta \log k, \delta \log k}$ subgraph has at most $k^{2-1/\delta \log k} \leq \varepsilon k^2$ edges

Result 1
An $\boldsymbol{f(k) \cdot n^{\Omega\left(\frac{k}{\log k}\right)}}$-time lower bound for constant approximating k-Clique would imply **PIH**.

Our Techniques



Reducing 3SAT to k-VectorSum

3SAT
Input: a CNF ϕ with n variables, m clauses
Goal: decide if ϕ is satisfiable

k-Vector Sum Problem
Input: $V_1, \ldots, V_k \subseteq \mathbb{F}^m$
Goal: Decide if $\exists u_1 \in V_1, \ldots, u_k \in V_k$, such that $\Sigma_{i \in [k]} u_i = 0$.

- **ETH**: 3SAT has no $2^{o(n)}$-time algorithm
- $N \triangleq |V_i| = 2^{O(n/k)}$
- **ETH** ⇒ k-Vector Sum has no $N^{o(k)}$-time algorithm

Main idea:
- WLOG, assume that each variable appear in at most 3 clauses
- Split m clauses into k groups with m/k clauses each
- vector set in V_i ⇔ assignment to i-th group **+** pairwise consistency bits for each variable

Pairwise consistency:
if $x_i = 0$, the corresponding coordinates are $0, 0$;
if $x_i = 1$, the corresponding coordinates are $1, -1$.

Reducing 3SAT to k-VectorSum

$$\underbrace{(x_1 \vee x_2 \vee x_3)} \wedge \cdots \wedge \underbrace{(\neg x_4 \vee x_5 \vee \neg x_2)} \wedge \cdots \wedge \underbrace{(x_6 \vee x_7 \vee x_8)} \wedge \cdots \wedge \underbrace{(\neg x_6 \vee \neg x_1 \vee x_2)}$$

k blocks

Pairwise consistency check!

satisfying partial assignment for block 1

		1	1						

$+$

satisfying partial assignment for block 2

		-1	0						

$=? \ \vec{0}$

entry for (x_2 ,block1, block2) … entry for (x_2 ,block1, block3)

…

$+$

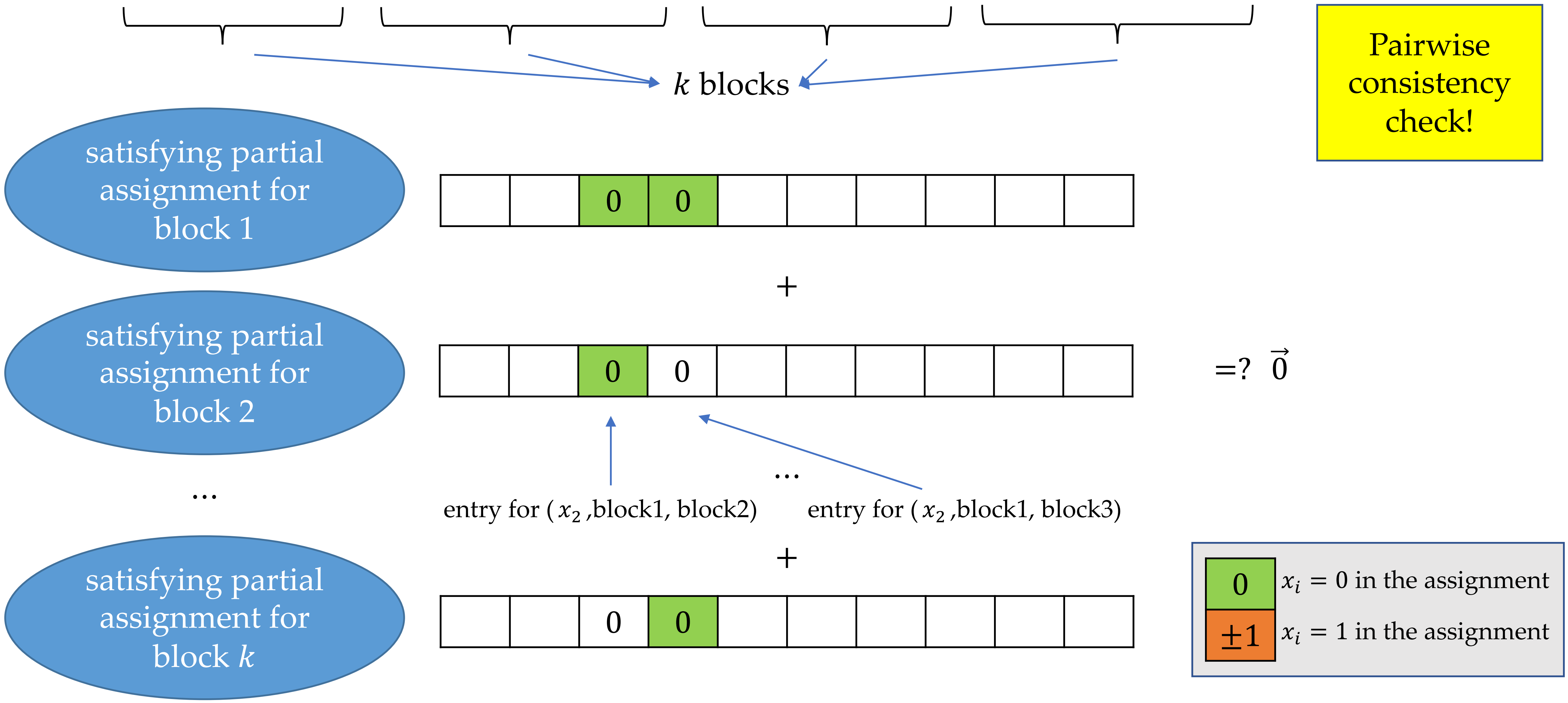
satisfying partial assignment for block k

		0	-1						

0	x_i = 0 in the assignment
± 1	x_i = 1 in the assignment

Reducing 3SAT to k-VectorSum

$$(x_1 \vee x_2 \vee x_3) \wedge \cdots \wedge (\neg x_4 \vee x_5 \vee \neg x_2) \wedge \cdots \wedge (x_6 \vee x_7 \vee x_8) \wedge \cdots \wedge (\neg x_6 \vee \neg x_1 \vee x_2)$$



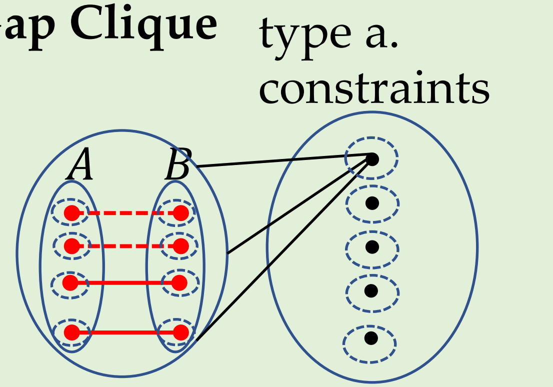
Reducing k-Vector Sum to Gap-Clique

k-Vector Sum Problem
Input: $V_1, \ldots, V_k \subseteq \mathbb{F}^m$
Goal: Decide if $\exists u_1 \in V_1, \ldots, u_k \in V_k$, such that $\Sigma_{i\in[k]} u_i = 0$.

Weak Gap-CSP
Variable set: **X**=$\{x_{a_1,\ldots,a_k} : a_1, \ldots, a_k \in \mathbb{F}\}$
Constraints:

a. Test if $x_{a_1,\ldots,a_k} = \sum a_i v_i$ for some unknown v_i's, $x_{b_1,\ldots,b_k} + x_{a_1,\ldots,a_k} = x_{a_1+b_1,\ldots,a_k+b_k}$
b. For all $i \in [k]$, test if $\exists u_i \in V_i$, $x_{b_1,\ldots,b_i+a,\ldots b_k} - x_{b_1,\ldots,b_k} = a \cdot u_i$
c. Test if $\Sigma_{i\in[k]} u_i = 0$, $x_{a+b_1,\ldots,a+b_k} - x_{b_1,\ldots,b_k} = 0$

Gap Clique



Lemma
(Yes) If k-Vector Sum has a solution $v_1 \in V_1, \ldots, v_k \in V_k$, then $x_{a_1,\ldots,a_k} = \sum a_i v_i$ satisfies all constraints.
(No) If k-Vector Sum has no solution, then for every assignment

- Either ε-fraction of the type a. constraints are not satisfied;
- Or $\exists$ a matching of variables s.t. ε-fraction of the matchings are not satisfied.

Dimension Reduction

k-Vector Sum Problem

Input: $V_1, \ldots, V_k \subseteq \mathbb{F}^m$

Goal: Decide if $\exists u_1 \in V_1, \ldots, u_k \in V_k$, such that $\Sigma_{i\in[k]} u_i = 0$.

Weak Gap-CSP

Variable set: $X = \{x_{a_1,\ldots,a_k} : a_1, \ldots, a_k \in \mathbb{F}\}$

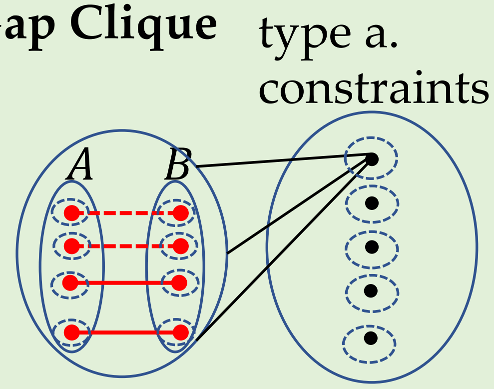
Constraints:

a. Test if $x_{a_1,\ldots,a_k} = \sum a_i v_i$ for some unknown v_i's, $x_{b_1,\ldots,b_k} + x_{a_1,\ldots,a_k} = x_{a_1+b_1,\ldots,a_k+b_k}$

b. For all $i \in [k]$, test if $\exists u_i \in V_i$, $x_{b_1,\ldots,b_i+a,\ldots b_k} - x_{b_1,\ldots,b_k} = a \cdot u_i$

c. Test if $\Sigma_{i\in[k]} u_i = 0$, $x_{a+b_1,\ldots,a+b_k} - x_{b_1,\ldots,b_k} = 0$

Gap Clique



Dimension reduction:

- $x_{a_1 \ldots a_k} \in \mathbb{F}^m$
- $m = k \log n$
- $|\mathbb{F}|^{k \log n}$ is too large
- Let $\ell = k + \log n$
- Pick $A_1, \ldots A_\ell \in \mathbb{F}^{k\times m}$ randomly
- Let $y_{\vec{\alpha},\vec{\beta}} = f(\vec{\alpha}, x_{\vec{\beta}}) \overset{\text{def}}{=} (\vec{\alpha} A_1 x_{\vec{\beta}}, \ldots, \vec{\alpha} A_l x_{\vec{\beta}}) \in \mathbb{F}^\ell$
- Add new constraints

New Constraints:

a. Test if $y_{\vec{\alpha},\vec{\beta}}$ is vector-valued degree-2 polynomial in terms of $\vec{\alpha}$ and $\vec{\beta}$.

b. Test if $y_{\vec{\alpha}+\vec{\gamma},\vec{\beta}} = y_{\vec{\alpha},\vec{\beta}} + y_{\vec{\gamma},\vec{\beta}}$ and $y_{\vec{\alpha},\vec{\beta}+\vec{\gamma}} = y_{\vec{\alpha},\vec{\beta}} + y_{\vec{\alpha},\vec{\gamma}}$.

c. For all $i \in [k]$, test if $y_{\vec{\alpha},\vec{\beta}+\vec{e_i}} - y_{\vec{\alpha},\vec{\beta}} = f(\vec{\alpha}, v_i)$, for some $v_i \in V_i$

d. Test if $y_{\vec{\alpha},\vec{\beta}+\vec{1}} - y_{\vec{\alpha},\vec{\beta}} = 0$

Conclusion

Our results:

- $\boldsymbol{n}^{\boldsymbol{\Omega}\left(\frac{\boldsymbol{k}}{\log \boldsymbol{k}}\right)}$-time lower bound for constant Gap k-Clique $\Rightarrow$ **PIH**
- **ETH** $\Rightarrow f(k) \cdot n^{\Omega(\log k)}$ -time lower bound for constant Gap k-Clique

Open problems:

- Improve lower bounds for constant Gap k-Clique
- $2^{o(n)}$-time lower bound for non-parameterized constant Gap Clique

Thanks for listening!